Algorithm Implementation of the Continuation Method for Solving a Static Equilibrium Problem of a Tensegrity Structure

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Firma Autor: José David Montoya

To all the people, I share my life with.

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ABSTRACT

Tensegrity structures are used in fields like architecture, engineering and biomedicine, and have new rising applications in robotics. The static analysis of these structures might be performed for the initial configuration of the structure in absence of external forces. This analysis leads to a set of equations generated from the Newtonian mechanic or the virtual work principle. Because this initial configuration of the structure is important, if different configurations can be found, it will lead to new applications and uses. There are numerical methods that search all the possible solutions in polynomial systems, one of them is the continuation method, that has been used in different fields such as biology and specially in robotics. In this work, two planar tensegrity structures are chosen to perform an analysis of static equilibrium using the virtual work principle, then a polynomial system is generated and solved with a computational implementation of the continuation method. The objective is to find different sets of solutions that might lead to new configurations of the tensegrity structures.

KEYWORDS:

Continuation method, tensegrity structures, polynomial system, static equilibrium.

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1. INTRODUCTION

1.1. TENSEGRITY STRUCTURES AND CONTINUATION METHOD

In engineering, polynomial systems arise naturally and the use and implementation of numerical methods to solve these systems have been very useful. There are specific problems that require for example solve large polynomial systems and obtain all the possible solutions. This specific approach is still a subject of research.

For solving polynomial system there are different methods: symbolic methods, numerical methods and geometric methods. Numerical methods become important when the polynomial system is non trivial, have high degree with many equations and variables. On the the other hand symbolic methods for example, from 25th degree and higher, could start to generate not so good results in the computational implementation [22].

Numerical methods used for solving polynomial systems are classified in iterative methods, like Newton's Method, that are good to find local solutions and require a good set of initial conditions; and continuation methods, that are appropriate for find all the possible solutions, but require more computational effort [22].

Polynomial systems are particularly attractive in problems like formula construction, geometric intersection problems, inverse kinematics, power flow problems with PQ-specified bases (a problem in which the real and reactive power are specified, and the voltage will be calculated), computation of equilibrium states, etc. For solving these systems the robustness of the method used is a concern. The methods must assure that all isolated solutions are obtained, an exhaustive method is wanted. The continuation method answers this characteristics [20].

The starting point of the continuation method consists in solve a simpler polynomial system and follow a path of solutions that will lead you to the original polynomial system and the solutions. The function that link the simpler polynomial system and the complex and wanted polynomial system is called homotopy function.

In engineering, tensegrity structures are used in several fields like, architecture, energy generation and aerospace. These structures have two principal components, rigid bodies or compressive parts and strings or tensile parts. A challenge that the engineers face when working with these structures, is the static equilibrium analysis, the tensegrity structures need to have a balance of forces to obtain a state of static equilibrium [43]. For solving this problems, the use of numerical methods is important for the correct design of the tensegrity structures.

The analysis of tensegrity structures with numerical methods is in continuous research, specially for robotics applications as noted for McCarthy [23]. Bayat and Crane [6] also have try to get more solutions for the static analysis of tensegrity structures. Jiang and Vijay [16] have conducted similar analysis of objects suspended from cables using the continuation method.

1.2. PROBLEM STATMENT

The numerical methods currently used in the state equilibrium analysis of tensegrity structures find only local solutions. This methods also require a good set of initial condition, which in some cases are difficult to known. The though of implementing a numerical method that will solve the static analysis of a tensegrity structure and obtain all the possible solutions, could give a lot of new configurations and be very useful for the engineers working with tensegrity structures. But is not simple because the static analysis will derive in systems of equations and these have to be transformed in polynomial systems. After the polynomial system is solved, the solutions should be mapped to the initial system and verify if they have physical meaning.

1.3. RESEARCH OBJECTIVES

The general objective of this work is to solve a static equilibrium problem for a tensegrity structure using the continuation method. This will be obtained through the implementation of an algorithm of the continuation method. This algorithm will solve a polynomial system derived from the static equilibrium analysis of a tensegrity structure. The algorithm implemented will be used for evaluate the influence of the parameters, variables and coefficients chosen for the continuation method.

1.4. SCOPE OF THE WORK

This first chapter is the introduction of the work. The second chapter will be focused on a brief state of art, explaining basic concepts of the tensegrity structures and the continuation method. In the third chapter the objectives of this work will be stated. The fourth chapter will explain the methodology and the way this work was done. The fifth chapter will be divided in a basic polynomial system to explain how the continuation method works highlighting the most important characteristics found during the implementation, then an analysis of the two tensegrity structures with the stated equations, the generated polynomial systems, the results, and the prove of static equilibrium of the solutions. The last chapter will be the conclusions and proposal for future work in this topic.

2. STATE OF ART

2.1. CONTINUATION METHOD

The continuation method was first introduced in the 80s, is a method that can find all solutions of polynomial systems with efficiency and reliability; the first applications used with the continuation method was solving kinematics problem [24]. The only limitation of the method is in the computational implementation, because it could require intensive computation. During two decades Sommense, Wampler, Morgan, Allgower and Li, published an exhaustive study of the method [26, 27, 28, 19, 2, 33]. The method was applied in different areas with special focus in robotics, in 1985 Tsai y Morgan [36] implemented an algorithm SYMPLE using the continuation method for solving the kinematics of the most general six and five degree of freedom manipulators. Su [34] accomplished an inverse static analysis of planar compliant mechanisms, Lee [17] used it for solving the geometric design problem of spatial 3R robot manipulators using polynomial homotopy continuation [17], Bin [7] used the continuation method applied to kinematics of parallel manipulator. For more examples see [35, 37, 41, 21, 25].

There are some software implementation of the continuation method in different programming languages like FORTRAN, C and MATLAB for solving polynomial systems. Verschelde implemented an algorithm of general purpose PHPack [39, 40], Lee et al implemented the HOM4ps2.0 [18], with an update HOM4ps3.0 [9] that use parallel implementation, Zang implemented a numeric toolbox NACLab [42] for MATLAB that include functionality to work with continuation methods. HOMOPACK developed by Irani et al [14], HOMOLAB [1] implemented in MATLAB which later will evolve and become Bertini[4] developed by Bates et al.

The software Bertini offers free and easy access to the latest advances in homotopy methods, features like use of adaptive multiprecision, which makes the computation robust and accurate, support of parallel computing allowing the user to take advantage of the availability of multicore technology in personal computers. Bertini can be used to produce all isolated solutions of a system of polynomials with complex coefficients. These points can be computed with up to several hundred digits of accuracy since Bertini makes use of multiple precision. The user does not need to know a priory the level of precision necessary to attain the desired accuracy because Bertini may be told to use adaptive precision changes. Another of Bertini's key capabilities is that it can find witness points on every irreducible component of the algebraic set, corresponding to a system of polynomials with complex coefficients. The bottom line is that Bertini will detect the irreducible decomposition of the algebraic set by specifying at least one point on each component. Once such a witness set is computed, the user may sample any component (i.e., produce as many points on the component as desired) and perform component membership (i.e., specify a list of points and ask Bertini to determine whether any of the points lie on the algebraic set corresponding to the witness set). Bertini allows the user to choose from several different types of runs and to specify a large number of tolerances (although all tolerances come with reasonable default settings)[3]. Also a book called Numerically Solving Polynomial Systems with Bertini [5] was written as a manual for helping new users to get started more quickly and illustrate the concepts of the polynomial continuation.

Li [19], explains how the continuation method works. The idea for solving a polynomial systems is to start with a simpler polynomial system that is easy to solve and to obtain all the solutions, and with the aid of a function, link it with the wanted polynomial system. This is done through a parameter t, which range is between 0 and 1, when the value of $t=0$, the polynomial system to solve is the simpler one, when the value of $t=1$ the polynomial system to solve is the desired one usually more complex. We show a specific case. The polynomial system we want to solve is

$$x^2 + y^2 = 5 \tag{1}$$

$$x - y = 1 \tag{2}$$

We start solving a simpler system,

$$x^2 = 1 \tag{3}$$

$$y = 1 \tag{4}$$

This system is easy to solve through inspection. Now we introduce the parameter t and link the systems in a function with the parameter t,

$$(1-t)\begin{bmatrix} x^2-1 \\ y-1 \end{bmatrix} + t\begin{bmatrix} x^2+y^2-5 \\ x-y-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

when in the system (5) *t=0*, we obtain the system (3)(4) and when *t=1* we have the system (1)(2), this type of functions are known as homotopy functions. The homotopy functions could have different structures or modifications according to the application or the implementation of the continuation method.

Some examples of homotopy functions are:

Polyhedral Homotopy

$$h_i(x,t) = \sum_{a \in A_i} c_a x^a t^{w_i(a)}, x^a = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}, t \in [0,1] \tag{6}$$

A polyhedral homotpy is induced by a lifting function $w = (w_1, w_2, \ldots, w_n)$ on the supports $A = (A_1, A_2, \ldots, A_n)$ of the polynomial system

Convex-Linear Homotopy

$$h_i(x,t) = (1-t)\gamma F^{(0)} + tF(x), \gamma \in \mathbb{C}, t \in [0,1] \tag{7}$$

The system $F^0(x) = 0$ is the start system.

Coefficient-Parameter Homotopy

$$h_i(x,t) = \sum_{a \in A_i} c_a(t) x^a, t \in [0,1] \tag{8}$$

Morgan explains in his book [26], two important characteristics of the continuation method: for each displacement of the t the method use the previous solution of the intermediate polynomial system and apply the newton method, smaller displacements of t more intermediate system will be solved. There will be more computational demand, but in some cases this will avoid the cross of the solutions paths.

The selection of the displacements of t, could have two approaches, a fixed step in which a number is chosen to be the number of equidistant steps. Morgan in the implementation starts with a step of 1/10000 and based on the solutions and the time it takes to solve the polynomial system the value could be modified. If non singular solutions start to repeat themselves, this could be the example of crossing paths, so a smaller t could avoid this behavior. A variable step is the other approach in which the step is given by other algorithm that usually employ the gradient of the functions to predict a good size of each step along each path, Morgan also shown an example of this [26].

Another important characteristic is, what is obtained in the continuation method is a path for each solutions that, start in *t=0* and end in *t=1*. In the end it will be easy to tell which solutions are general, singular, infinite solutions or solutions to infinite, the visual representation is very helpful to discover the solutions that exists at the end of the path which are the solution that have physical meaning. In Figure 1 there are two paths, each solution generate a path though the t axis with real and imaginary part. Also in Figure 2 the projection of the solution is shown in the t axis

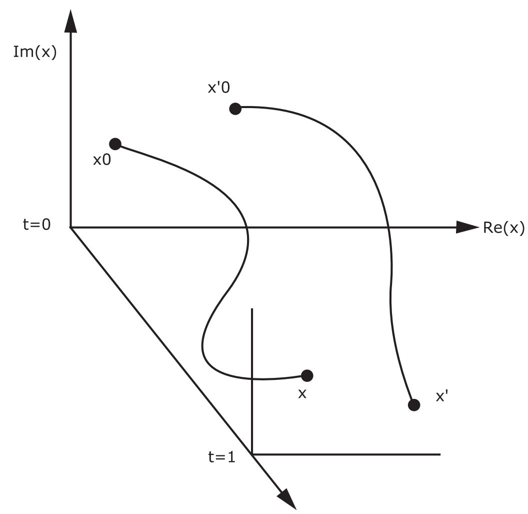

Figure 1. Two Solutions Path By The Continuation Method [26]

2.2. TENSEGRITY SYSTEMS

The term tensegrity was created by Richard Buckminster Fuller as a contraction of tensional and integrity[12]. Tensegrity structures are more that 60 years old, they started as an art manifestation and are currently used in many different fields like engineering, biomedicine and mathematics. Tensegrity structures are similar in appearance to the normal bar-joint structure, but in their static equilibrium there are presents forces within the structure that helps to balance and to maintain their shape, without this balance of forces the structures becomes unstable, this is a challenge in the design of the tensegrity structures and is called

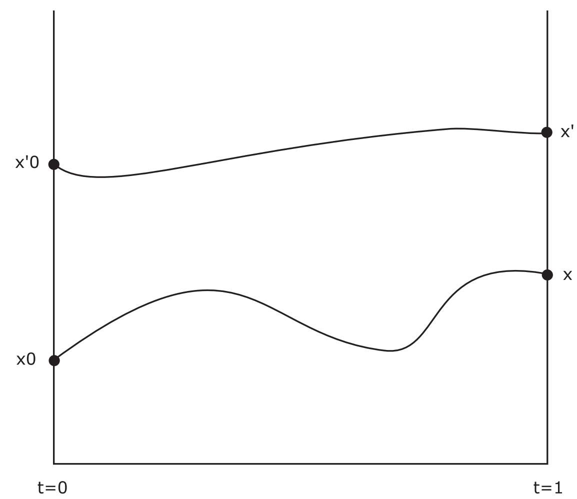

Figure 2. Path Projection Of The Solutions By Continuation Method [26]

form-finding or shape finding [43]. The tensegrity structures have compressive parts and tensile parts. The compressive parts are simply rigid bodies, and the tensile parts simply strings.

Characteristics of a tensegrity structure:

- The structure is free-standing, without any support.
- The structural members are straight.
- There are only two different types of structural members: struts carrying compression and cables carrying tension.

In Art tensegrity structures are used because tensegrity components are very simple elements, often just straight lines. This type of beauty has appealed to sculptors like Kenneth Snelson [32] and architects like Buckminister Fuller [11]. Figure 3 shown a physical construction of a tensegrity structure.

In nature, the bones and tendons of animals are connected in a way that allows easy control of movement. Obviously, these structures evolved for control functions, where the bones provide compressive load-carrying capacity and the tendons provide the stabilizing tensions required in a given configuration. This structures are similar to a tensegrity structure and could be modeled by one.[31]

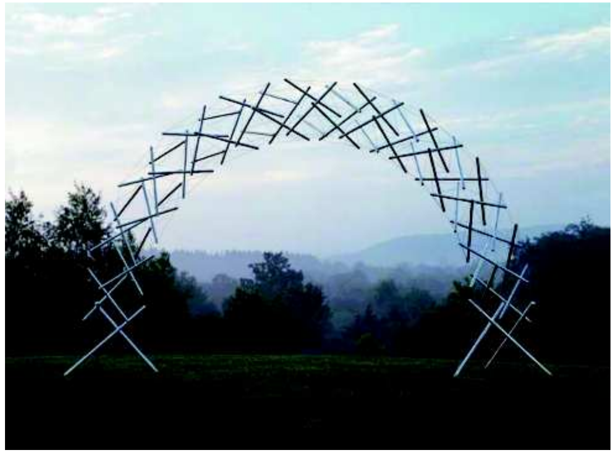

Figure 3. Tensegrity sculpture built by Kenneth Snelson [31]

The simplest tensegrity structure is a single rigid body and a string. In the case of bars and strings, this is simply a prestressed bar [31]. The next simplest, and first nontrivial tensegrity structure with bars and strings, is two bars and four strings, the next simplest tensegrity structure is shown in Figure 4 is a fundamental three dimensional unit we will call a tensegrity prism [31].

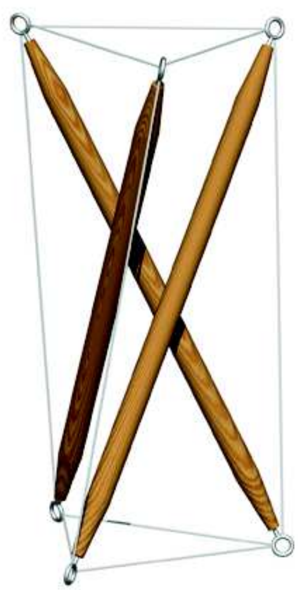

Figure 4. Tensegrity Prism [8]

Tensegrity structures have been studied and applied in a several fields, like civil engineering

[13] , aerospace [10], wave energy harvesting [38], human anatomy [29], modeling the motion viruses [30] and aquaculture [15].

3. OBJECTIVES

3.1. GENERAL OBJECTIVE

Solve a static equilibrium problem for a tensegrity structure using the continuation method

3.2. SPECIFIC OBJECTIVES

- To define the tensegrity structure that will be used for the analysis
- To formulate a mathematical model that represents the tensegrity structure, this will be a polynomial system.
- To develop a code from the mathematical model, using a numerical approach based on the continuation method for solving the polynomial system
- To evaluate the influence of the parameters and variations of the continuation method that will be used for solving the system
- To design a graphical interface, for enter the parameters and shown the solution of the polynomial system

4. METHODOLOGY

The first step was a review related with the solution of polynomial systems using the continuation method, the implementation of this method in different programming languages and the basics of tensegrity structures. The scientific data bases consulted were Science Direct, ASME, Springer, IEEE. Also attended to the Workshop on Software and Applications of Numerical Algebraic Geometry at the University of Notre Dam and establish contact with Andrew Sommese, Charles Wampler, and Daniel Bates, researchers that have been working in the continuation method for years.

After the compilation of the information, the focus was in understanding the continuation method, for this, the implementation of small and basic polynomial systems was done, and then incremented the complexity until the knowledge of the method, allows to implement more general polynomial systems. The guidance of the book of Alexander Morgan [26] was key to understand and do the implementation of the method in MATLAB.

Two tensegrity structures were defined. The first structure consists in one rigid body and two tensile parts fixed in two struts. In the implementation the length of the rigid body can be changed, also the distant between the struts and the origin, but the first strut can only have value on the x axis (a_0) and the second strut can only have a value on the y axis (b_0). This structure is not considered as a real tensegrity structure but is used as a first step for implementing the polynomial continuation algorithm. This first structure help to understand the approach of generate a polynomial system, to established the workflow and to implement a non to intensive algorithm. The second structure is known as a planar tensegrity structure consists in two rigid bodies and three tensile parts, the tensile parts connect the rigid bodies. As a simplification, the rigid bodies have the same length. This length can be changed, and the distant between the origin of each rigid body, but always as part of the x axis.

Then a static equilibrium analysis using the virtual work principle was performed for each structure. This is an energy approach which permits to obtain the equilibrium equations. These equations will be modified to get a polynomial system. This process included adding new variables to eliminate the square roots and replacing the trigonometrical functions using the half angle formulas.

Then an algorithm of the continuation method was implemented for solving the each polynomial system. This include a search of parameters that will allow to obtain all the possible solutions.The homotopy function used is the one implemented in other software like Bertini and CONSOL. This homotopy function is very intuitive, easy to scale and implement, also provides a good performance as will allow to find all the possible solutions to the polynomial system. Also the projective transform was implemented to verify the results. An extra verification was made using the software Bertini, validating all the results obtained in this implementation. The solutions found will be discussed and the singularities. The choice of the parameter t will be analyzed with different values, and a proposed strategy for this choice will be made.

A user interface was built, to facilitate the change of some of the parameters of the implemented algorithm and to see the results of the static equilibrium analysis of the two tensegrity structures.

5. RESULTS AND DISCUSSION

5.1. SOLVING A BASIC POLYNOMIAL SYSTEM

The first step in the process was to understand the method, starting with the implementation of simple polynomials systems, following the bibliography. Most of the sources describe the method in a general form and use it to solve specific problems, but a more exhaustive explanation was needed. Alexander Morgan wrote a book [26] Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems, a key part of this process, because is focused in the implementation and the mathematical theory behind it, he also shows in his book the code in FORTRAN of a prototype of various solvers using the continuation method theory, with this guideline the method was implemented for a simple polynomial system with no physical meaning and for two polynomial systems generated from tensegrity structures.

It is important to empathize the key elements of the method, based in the experience of the implementation of the algorithms.

- Build a simple starting system for our polynomial system, it must have the same degree of the original. This system should be very easy to solve, its solutions must be independent and non singular. These characteristics are obtained by selecting a simple form of the polynomial system, having random constants as coefficients. Morgan [26] explains that this independence will allow to find all the solutions of the original polynomial system. In the implementation of the algorithm some of the constant shown in the book are used, but random numbers generated in MATLAB are also used. These numbers are complex with a range between -1 and 1 and without dependence to each other.
- After the formulation of the starting system, an expression is created to link the polynomial to be solved and the starting system, this expression is called homotopy function. The are different choices for this function but only one is implemented. The general concept is to introduce a variable t, which will link the two systems, and this variable t will variate for each step in the iteration process of the method, becoming the starting

system when t = 0 and the original system when t=1. In this implementation, delta t is constant, and the used values were between 0.001 and 0.00001, this choice will let you find all the paths, and avoid the crossing paths, but also will have a consequence of the computational time required to execute the code. The general criteria for choosing t, its to avoid path crossing. If for example the algorithm find two or more identically non singular solutions this indicate a path crossing. This will lead to try a step size smaller. Another scenario its to not find the conjugate of complex solutions, this will indicate a crossing paths with a solution at infinite. Usually the starting t is 0.0001 which is the one proposed for Morgan [26] and adjusted for avoiding path crossing and to decrease computational time in each implementation. Morgan [26] also presents another kind of solvers for choosing delta in a more adaptive way to the path, a variable step algorithm, improving the computational time and still finding all the solutions of the polynomial systems. Those were not implemented in this work.

- For each solution of the start system a solution of the original system is tracked incrementing t and solving the equations using the Newton's method; using the determinants or using the linear combination and getting the residue, there is a convergence tolerance that will affect each step of each path, in general, the residue must be less than 0.0001.
- The paths that do not converge are called solutions at infinite and have the tendency to acquire values with very high magnitude. When this behavior is presented, the algorithm will classify them as a solution to infinite, and will pass to the other path. This behavior as Morgan [26] stated is presented at the end of path with values of t very close to 1.
- In the implementation of the Newton's method in each step and each iteration can present converging problems. A tolerance is set of 100 iteration maximum. This is actually very high because of nature of t, usually it just take between 1 to 5 iteration to converge in each step of the path.

First a simple polynomial system will be solved, to illustrate and explain the process of solving a polynomial system with the continuation method.

For the selection of a polynomial system, one stated in [26] is used, is a polynomial system of two equations in two variables:

$$x_1x_2 - 6x_2 + 11x_1 - 6 = 0 \tag{9}$$

$$x_1^2 - x_2 = 0 \tag{10}$$

The implementation begin with the creation of the starting system, the degree of each polynomial is 2, which is equal of the maximum degree found in all the individual terms, with a total degree of 4. Then complex constants are created for this starting systems, the choice of this variables should be random, this will guarantee the non linear dependence between them and will satisfy the independence principle that makes the method work, this have a mathematical explanation that can be found in [26]. The starting system also have to be chosen to be very simple and easy to scale when higher degree system start to arise, the structure of the starting system used is:

$$g_j(x) = p_j^{d_j} x_j^{d_j} - q_j^{d_j} \tag{11}$$

Where j=1,2,..., n. d_j is the degree of the polynomial, p_j and q_j are the complex constant. x_j is the variable of the polynomial system.
The two polynomial systems linked by the homotopy function will look like this:

$$(1-t)\begin{bmatrix} p_1^2 x_1^2 - q_1^2 \\ p_2^2 x_2^2 - q_2 \end{bmatrix} + t\begin{bmatrix} x_1x_2 - 6x_2 + 11x_1 - 6 \\ x_1^2 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

Using proposed constants which are:
p_1=0.12324754231 + 0.76253746298i
p_2=0.93857838950 - 0.99375892810i
q_1=0.58720452864 + 0.01321964722i
q_2=0.97884134700 - 0.14433009712i

The solutions found are: (1,1),(2,4) and (3,9). The steps of the algorithm implemented are shown in Figure 5.

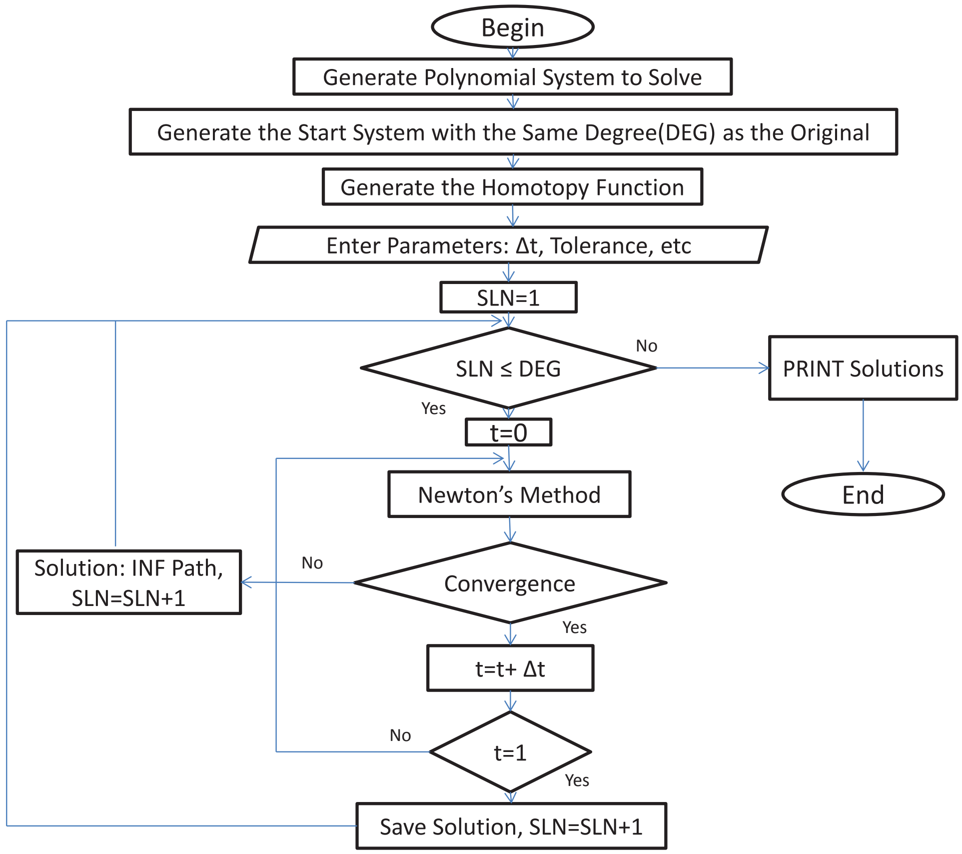

Figure 5. Algorithm

There is one solution missing because the total degree of the system which is the multiplication of the degree of each polynomial is 4. There are 2 options, one of the solutions has a multiplicity greater that 1, that is an indicator of a singular solution. The way to prove this is looking at the determinant and to evaluate it with the solution if the result is nonzero, then the solution is not singular. The other case is when a solution is at infinity. The way to find if a solution is at infinite is using the projective transformation [26], this transformation can be associated with a change of base in linear algebra.

To calculate the projective transformation the first step is the homogenization of the system, adding a variable to each term to match the maximum degree of the polynomial, to avoid confusion a change in the nomenclature is made, y instead of x. For example the first therm of the first equation 9 is x_1x_2, the homogenization of this therm is simple a change of variable because the degree is equal to the maximum of the polynomial system, as a result the homogenization of this therm is y_1y_2. The second therm of 9 is $6x_2$, degree equal to 1, as stated previously the homogenization have to be the maximum degree. Then the new variable y_3 is included to fulfill this requirement, obtaining the homogenization $6y_2y_3$. The new variable should have the degree necessary to match maximum degree of the polynomial. After this process the polynomial system obtained is:

$$y_1y_2 - 6y_2y_3 + 11y_1y_3 - 6y_3^2 = 5 \tag{13}$$

$$y_1^2 - y_2y_3 = 0 \tag{14}$$

This lead to an inconsistent polynomial system of 3 variables and 2 unknowns, this is when y_3 have to be replaced by a therm that allows the transformation between the two polynomial system. This therm can have any value, for example if is a constant equal to 1, the initial polynomial system is obtained. But a more general system is wanted, one that all solutions can be followed through the paths. This is achieved by $L(x)$, a linear equation where the constant are random numbers, and the variables are given based on the number of variables of the original system.

$$L(x) = a_1y_1 + a_2y_2 + a_3 \tag{15}$$

$L(x)$ is important because will allow the transformation of the solutions from the homogenized to the original polynomial system. Replacing y_3 for $L(x)$ will help to make the polynomial consistent again, in this case is a linear combination of the variables y_1 and y_2. Three new constant a_1, a_2, a_2 are added, this are random constant and have the same characteristics: random numbers that preserve the independence principle, Alexander Morgan [26] gave an example of these numbers. a_1=0.13782974, $a1$=-0.89225233, a_3=0.11230581.

$$y_3 = L(x) \tag{16}$$

But this can be changed, for example another of set of three random numbers are chosen, this will lead to a new polynomial system, the idea is to show that the choice of the numbers of $L(x)$ does not change the final results of the polynomial system, these numbers does not have to be complex, because the coefficients of the polynomial are real numbers, this exercise was also solved with these other constants a_1=1, a_2=5, a_3=-8. After this transformation, the new polynomial system is generated, which is called the projective transformation:

$$
\begin{aligned}
Eq1 &= A_1x^2 + B_1y^2 + C_1xy + D_1x + E_1y + F_1 \\
A_1 &= 11a_1 - 6a_1^2 \\
B_1 &= -6a_2 - 6a_2^2 \\
C_1 &= 1 - 6*a_1 + 11*a_2 - 12*a_1*a_2 \\
D_1 &= 11*a_3 - 12*a_1*a_3 \\
E_1 &= -6*a_3 - 12*a_2*a_3 \\
F_1 &= -6*a3^2
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
Eq2 &= A_2x^2 + B_2y^2 + C_2xy + D_2x + E_2y + F_2 \\
A_2 &= 1 \\
B_2 &= -a_2 \\
C_2 &= -a_1 \\
D_2 &= 0 \\
E_2 &= -a_3 \\
F_2 &= 0
\end{aligned} \tag{18}
$$

The results of this polynomial system will be different depending of the choice of the constants, but all the solutions of this system correspond to the solutions of the original system. To obtain the solution of the original polynomial system the projective transformation should be divided by $L(x)$, which mean each finite solution will have a finite solution in the other system, and a solution at infinite in the original polynomial system will have also correspond finite solution in the projective transform. When doing the transformation (solution divided by $L(x)$), $L(x)$ will have a value of 0, becoming a solution to infinite in the original polynomial system.

Solving the projective transform applying the continuation method, 4 solutions were obtained:
(0.064012975274184,0.064012975657866),
(0.052316169976961,0.104632339953923),
(0.039100146641175,0.117300439923525) and
(2.104935e-10 + 7.716731e-10i, 0.125867768785346 - 0.000000000125303i)

Then $L(x)$ is calculated, to obtain the solutions to the initial polynomial system, one of the

$L(x)$ have very small values 5.838583783113194e-11 + 2.181614317141776e-10i, which is considered 0, so this is a solution to infinite. The other three solutions are traced back and consist on the ones discovered previously: (1,1),(2,4) and (3,9). As stated before this example was solved with different sets of numbers in $L(X)$, if the numbers where independent, the same results were obtained. Figure 6. shows the workflow for the implementation of the projective transform.

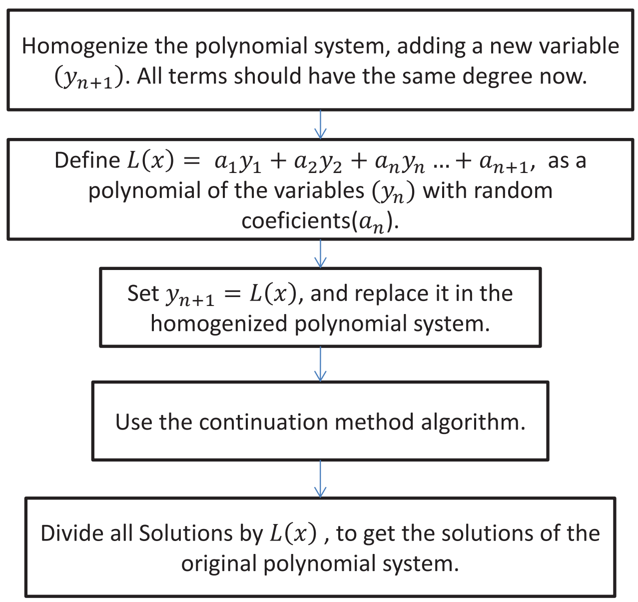

Figure 6. Projective Transform Implementation Steps

5.2. FIRST TENSEGRITY STRUCTURE

For the static equilibrium analysis of a basic structure, a system of equations can be generated and then with some substitutions transform it into a polynomial system. This structure is not strictly a tensegrity structure, but will allow to introduce the concepts and workflow necessary to perform the static equilibrium analysis and the implementation of the continuation method. For simplicity this first structure will be refereed as a tensegrity structure. This first planar tensegrity structure consist of a solid body and two tensors, modeled as springs, see Figure 7. Table 1 shows the characteristics of the elements and nomenclature used.

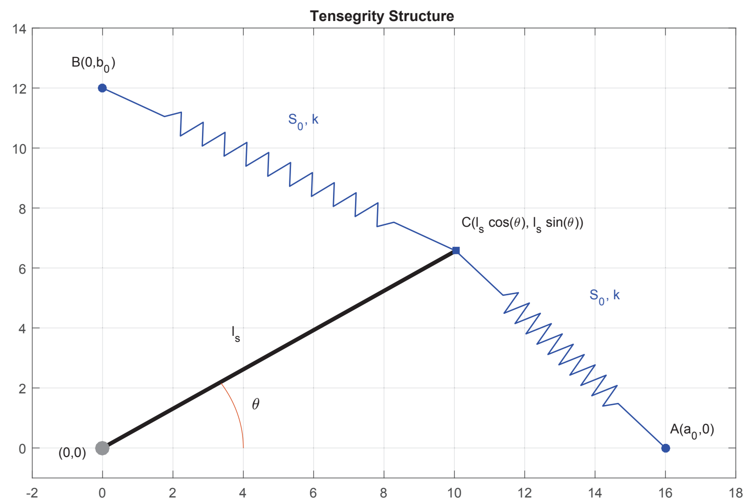

Figure 7. Tensegrity Structure Example 1

Table 1. Characteristics of the first tensegrity structure

Characteristic	Nomenclature	Additional information
The origin	(0,0)	Fixed pivot of the rigid body, cannot take other values
Location first strut	$A(a_0,0)$	a_0 can take other positive values
Location second strut	$B(0,b_0)$	b_0 can take other positive values
Length of the rigid body	l_s	l_s can take other positive values
Length of each tensor	S_0	S_0 can take other positive values
Stiffness of each tensor	k	k can take other positive values
Location of the end of the rigid body	$C(l_sCos(\theta),l_sSin(\theta))$	Its value is function of the angle θ

The potential energy stored in each spring is given by

$$V_i = \frac{1}{2}k_i(S_i - S_{0i})^2 \tag{19}$$

Where S_i is the actual length of the spring. For Example, for the lower spring.

$$S_1 = \sqrt{(C_x - a_0)^2 + C_y^2} \tag{20}$$

Where $C_x = l_sCos\theta$ and $C_y = l_sSin\theta$

When the contribution to the potential energy provided by both springs are annulled, the resultant expression is function of θ. This is

$$V(\theta) = V_1 + V_2 \tag{21}$$

The Principle of virtual work establishes that

$$\frac{\partial V}{\partial \theta} = 0 \tag{22}$$

After following the procedure described here, equation 23 is obtained

$$F = l_s a_0 sin(\theta) - l_s b_0 cos(\theta) - \frac{l_s s_0 a_0 sin(\theta)}{\sqrt{l_s^2 - 2cos(\theta) l_s a_0 + a_0^2}} + \frac{l_s s_0 b_0 cos(\theta)}{\sqrt{(l_s^2 - 2sin(\theta) l_s b_0 + b_0^2)}} \tag{23}$$

In equation 23 there are square roots in the denominator and also sines and cosines. Denominators can be eliminated by defining D_1 and D_2 as:

$$D_1 = \sqrt{l_s^2 - 2cos(\theta) l_s a_0 + a_0^2} \tag{24}$$

$$D_2 = \sqrt{l_s^2 - 2sin(\theta) l_s b_0 + b_0^2} \tag{25}$$

Multiplying equation 23 by $D_1 D_2$ and substituting

$$sin(\theta) = \frac{2x}{1+x^2} \tag{26}$$

$$cos(\theta) = \frac{1-x^2}{1+x^2} \tag{27}$$

where

$$X = tan(\frac{\theta}{2}) \tag{28}$$

The following equation results

$$\begin{aligned} eqF = {} & D_1 l_s s_0 b_0 k_0 - D_1 D_2 l_s b_0 k_0 + 2 D_1 D_2 l_s a_0 k_0 X - 2 D_2 l_s s_0 a_0 k_0 X + D_1 D_2 l_s b_0 k_0 X^2 \\ & - D_1 l_s s_0 b_0 k_0 X^2 \end{aligned} \tag{29}$$

If expressions for D_1 and D_2 are squared and the same substitutions for $sin(\theta)$ and $cos(\theta)$ are applied, the result is

$$eqD1 = D_1^2 X^2 + D_1^2 - l_s^2 X^2 - l_s^2 - 2 l_s a_0 X^2 + 2 l_s a_0 - a_0^2 X^2 - a_0^2 \tag{30}$$

$$eqD2 = D_2^2 X^2 + D_2^2 - ls^2 X^2 - ls^2 + 4 l_s b_0 X - b_0^2 X^2 - b_0^2 \tag{31}$$

A set of values are given for the constants, for the purpose of testing numerically the equations, the constant chosen are l_s=12, S_0=2, a_0=16, b_0=10, k_0=1. In the process of applying the method, the constant were evaluated and the variables renamed, for a clearer visualization of the equations; $x_1 = X$, $x_2 = D_1$, $x_3 = D_2$. Ending up with the polynomial system:

$$eq1 = 240x_2 - 120x_2x_3 + 384x_1x_2x_3 - 768x_1x_3 + 120x_1^2x_2x_3 - 240x_1^2x_2 \tag{32}$$

$$eq2 = x_1^2x_2^2 + x_2^2 - 784x_1^2 - 16 = 0 \tag{33}$$

$$eq3 = x_3^2x_1^2 + x_3^2 - 244x_1^2 + 480x_1 - 244 \tag{34}$$

The degree of each equation is 4, therefore the number of solutions will be 64. The workflow is the same as used in the first example, but now some of the tolerances should be tougher because some of the paths start to cross, and is not a desired behavior, but this also start to become a problem because the computational time starts to increase. After the implementation 16 solutions were found, this mean 48 are solutions to infinite. The 16 solutions found are not singular solutions and 8 of them reals, these are the ones that make more sense for a physical problem and are shown in Table 2. This solutions were verified and validated using the projective transform and Bertini. Bertini took less than 10 seconds to find the same set of solutions, the algorithm implemented took about 10 minutes. Bertini find solutions with about 15 significant digits, the algorithm implemented have 11 significant digits. With these

Table 2. Real Solutions First System

Real Solutions	X	D_1	D_2
1	0.19378	-6.6179	12.4240
2	0.29825	8.8733	10.608
3	0.39495	10.938	-8.9446
4	0.2755	-8.3772	-11.004
5	-3.0178	26.609	-19.680
6	-3.409	-26.891	-19.330
7	-3.5865	26.993	19.188
8	-4.0879	-27.215	18.836

solutions we have to obtain the original unknown θ transforming back the variables. Real values need to be checked if they satisfy the static equilibrium condition. Since the strut is a member under two forces, one at the fixed and the other at the free end, both forces must be equal in magnitude but with opposite directions. In other words, the unit vector of the spring forces acting on the free end of the strut must be in the opposite direction of the unit vector that goes form the fixed end to the free end. Therefore the scalar product between these unit vectors must be -1 in the equilibrium, see Figure 8 for the free body diagram.

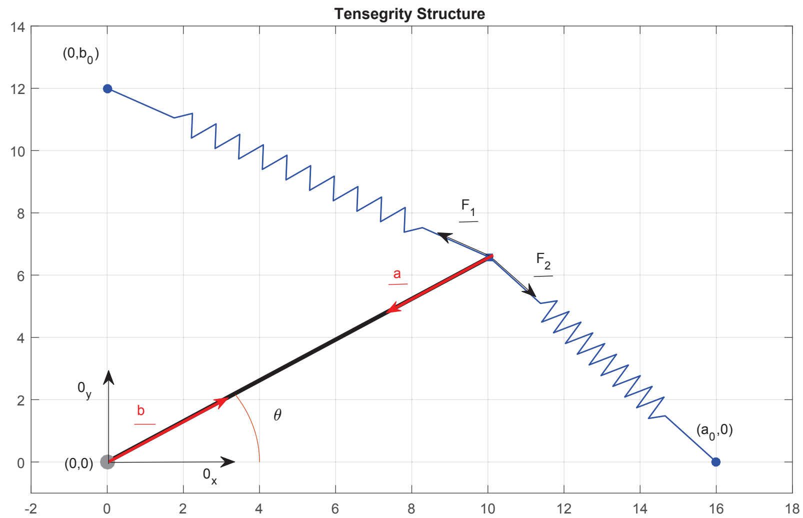

Figure 8. Free Body Diagram Example 1

For this calculation the only variable needed is the angle. It is worth to mention that since variables D_1 and D_2 are distances they can not be negative. This will be to took in to account later. But for now with each value of the angle, the scalar product is calculated, that is the way of prove static equilibrium of this tensegrity structure. Table 3 show all the comprobations, solutions 2 and 7, fulfills with the static equilibrium condition exactly and are illustrated in Figure 9 and 10. Solutions 5, 6 and 8 are close to -1 but the distances D_1 and D_2 are negative, because of that, only the solutions 2 and 7 will be accepted as a real solutions for the tensegrity structure.

Table 3. Comprobation of Static Equilibrium of Real Solutions First System

Real Solutions	X	$\theta(rad)$	$ScalarProduct$
1	0.19378	0.3828	-0.8078
2	0.29825	0.5797	-1.0000
3	0.39495	0.7523	-0.7945
4	0.2755	0.5377	-0.9835
5	-3.0178	-2.5016	-0.9991
6	-3.409	-3.4090	-0.9999
7	-3.5865	-2.5978	-1.0000
8	-4.0879	-2.6618	-0.9996

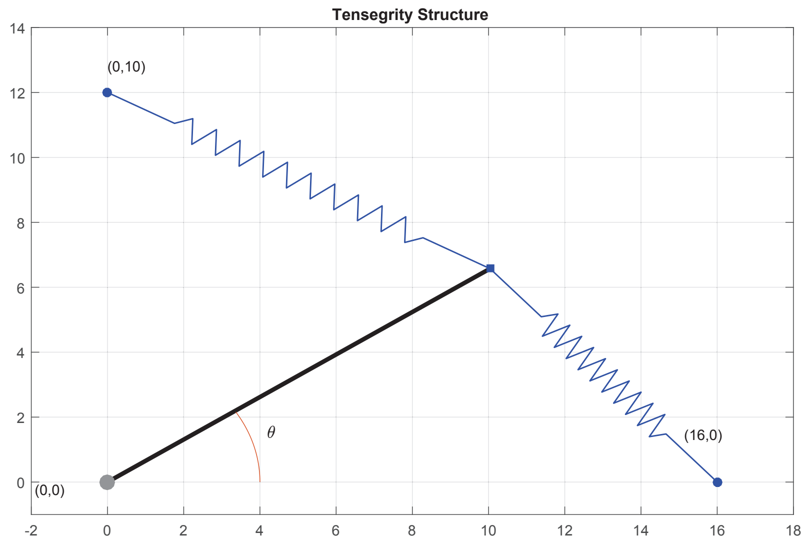

Figure 9. Tensegrity Structure Example 1 Solution 2

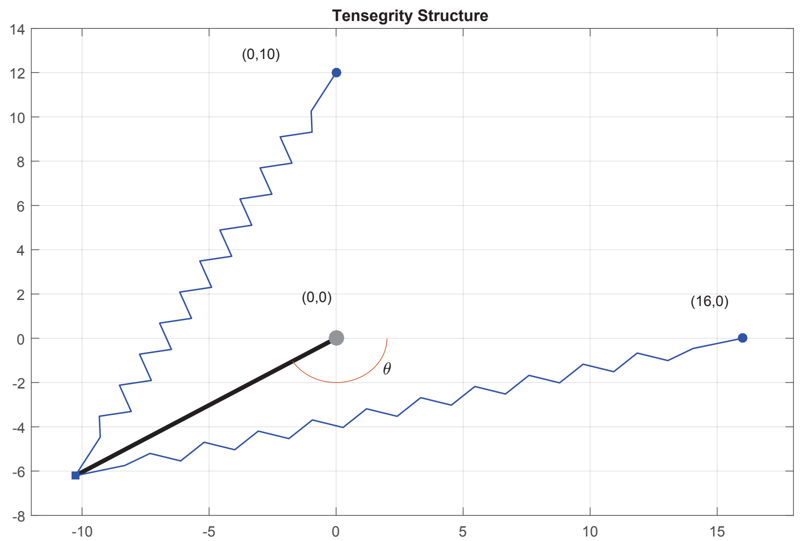

Figure 10. Tensegrity Structure Example 1 Solution 7

5.3. SECOND TENSEGRITY STRUCTURE

Another planar tensegrity structure was chosen, an equation system was derived using the virtual work principle and then transform it to a polynomial system, this tensegrity structure consist of two rigid bodies, each with a fixed pivot in (0,0) and (e_0,0) and three tensors connecting each strut as shown in Figure 11, as before the tensors are modeled as springs. Table 4 shows the characteristics of the elements and nomenclature used.

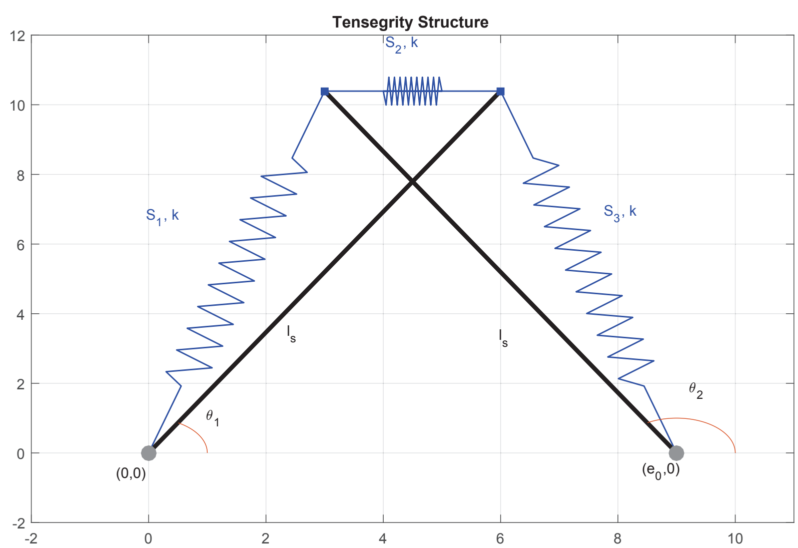

Figure 11. Tensegrity Structure Example 2

Table 4. Characteristics of the Second Tensegrity Structure

Characteristic	Nomenclature	Additional information
The origin of the first rigid body	(0,0)	Fixed pivot of the rigid body, cannot take other values
The origin of the second rigid body	$(e_0,0)$	Fixed pivot of the rigid body, e_0 can take other values
First Tensor	S_1	Located between (0,0) and the free end of the second rigid body
Second Tensor	S_2	Located between the free end of both rigid bodies
Third Tensor	S_3	Located between $(e_0,0)$ and the free end of the first rigid body
Length of the rigid bodies	l_s	l_s can take other positive values, but is the same for the two rigid bodies
Length of each tensor	S_0	S_0 can take other positive values, but is the same for all the tensors
Stiffness of each tensor	k	k can take other positive values, but is the same for all the tensors
Angle between the first rigid body and the x axis	θ_1	-
Angle between the second rigid body and the x axis	θ_2	-

The equilibrium equations are found using the principle of virtual work:

$$\begin{aligned}F(1) = {} & l_s^2 cos(\theta_1) sin(\theta_1) - l_s^2 cos(\theta_1) sin(\theta_2) + 2e_0 l_s sin(\theta_1) - \frac{e_0 l_s s_0 sin(\theta_1)}{\sqrt{e_0^2 - 2e_0 l_s cos(\theta_1) + l_s^2}} \\ & + \frac{l_s^2 s_0 cos(\theta_1) sin(\theta_2) - l_s^2 s_0 cos(\theta_2) sin(\theta_1) - e_0 l_0 s_0 sin(\theta_1)}{\sqrt{Aux_1}}\end{aligned} \tag{35}$$

$$\begin{aligned}F(2) = {} & l_s^2 cos(\theta_1) sin(\theta_2) - l_s^2 cos(\theta_2) sin(\theta_1) - 2e_0 l_s sin(\theta_2) \\ & + \frac{e_0 l_s s_0 sin(\theta_2)}{\sqrt{e_0^2 + 2cos(\theta_2) e_0 l_s + l_s^2}} \\ & + \frac{-(l_s^2 s_0 cos(\theta_1) sin(\theta_2)) + l_s^2 s_0 cos(\theta_2) sin(\theta_1) + e_0 l_s s_0 sin(\theta_2)}{\sqrt{Aux_1}}\end{aligned} \tag{36}$$

where
$Aux_1 = e_0^2 - 2e_0 l_s cos(\theta_1) + 2e_0 l_s cos(\theta_2) + l_s^2 cos(\theta_1)^2 - 2l_s^2 cos(\theta_1) cos(\theta_2) + l_s^2 cos(\theta_2)^2$
$+ l_s^2 sin(\theta_1)^2 - 2l_s^2 sin(\theta_1) sin(\theta_2) + l_s^2 sin(\theta_2)^2$
The previous equations have trigonometric functions, and square roots, with a variable change a polynomial system can be generated following the guidelines presented in the previous

example. As a result five polynomials in the variables D_1, D_2, D_3, X_1 and X_2 are obtained.

$$\begin{aligned} Poly1 = {} & 2D_2D_3l_s^2k_0X_1 - 2D_2D_3l_s^2k_0X_2 - 2D_2l_s^2s_0k_0X_1 + 2D_2l_s^2s_0k_0X_2 \\ & - 2D_2D_3l_s^2k_0X_1X_2^2 + 2D_2D_3l_s^2k_0X_1^2X_2 + 4D_2D_3e_0l_sk_0X_1 \\ & + 2D_2l_s^2s_0k_0X_1X_2^2 - 2D_2l_s^2s_0k_0X_1^2X_2 - 2D_2e_0l_ss_0k_0X_1 \\ & - 2D_3e_0l_ss_0k_0X_1 + 4D_2D_3e_0l_sk_0X_1X_2^2 - 2D_2e_0l_ss_0k_0X_1X_2^2 \\ & - 2D_3e_0l_ss_0k_0X_1X_2^2 \end{aligned} \tag{37}$$

$$\begin{aligned} Poly2 = {} & 2D_1D_3l_s^2k_0X_2 - 2D_1D_3l_s^2k_0X_1 + 2D_1l_s^2s_0k_0X_1 - 2D_1l_s^2s_0k_0X_2 \\ & + 2D_1D_3l_s^2k_0X_1X_2^2 - 2D_1D_3l_s^2k_0X_1^2X_2 - 4D_1D_3e_0l_sk_0X_2 \\ & - 2D_1l_s^2s_0k_0X_1X_2^2 + 2D_1l_s^2s_0k_0X_1^2X_2 + 2D_1e_0l_ss_0k_0X_2 \\ & + 2D_3e_0l_ss_0k_0X_2 - 4D_1D_3e_0l_sk_0X_1^2X_2 + 2D_1e_0l_ss_0k_0X_1^2X_2 \\ & + 2D_3e_0l_ss_0k_0X_1^2X_2 \end{aligned} \tag{38}$$

$$Poly3 = D_1^2X_2^2 + D_1^2 - e_0^2X_2^2 - e_0^2 + 2e_0l_sX_2^2 - 2e_0l_s - l_s^2X_2^2 - l_s^2 \tag{39}$$

$$Poly4 = D_2^2X_1^2 + D_2^2 - e_0^2X_1^2 - e_0^2 - 2e_0l_sX_1^2 + 2e_0l_s - l_s^2X_1^2 - l_s^2 \tag{40}$$

$$\begin{aligned} Poly5 = {} & D_3^2X_1^2X_2^2 + D_3^2X_1^2 + D_3^2X_2^2 + D_3^2 - e_0^2X_1^2X_2^2 - e_0^2X_1^2 \\ & - e_0^2X_2^2 - e_0^2 - 4e_0l_sX_1^2 + 4e_0l_sX_2^2 - 4l_s^2X_1^2 + 8l_s^2X_1X_2 - 4l_s^2X_2^2 \end{aligned} \tag{41}$$

This polynomial system was implemented using the continuation method, with the objective to finding all the solutions that satisfy the polynomial system, the original system and have physical meaning. A set of constants with a predefined value where chosen: e_0=9, l_s=12, s_0=2, k_0=1, to implement this polynomial system numerically.

This polynomial system have degree of 2400, which mean 2400 paths have to be tracked and find equal number of solutions. After the implementation 108 possible solutions where found, 30 of them reals and are shown in Table 5. Singular solutions were found in the subset of real solutions, 8 in total. The solutions 23 to 30 in Table 5 are the singular solutions all these solutions have multiplicity of 2. The rest of the solutions (2284) are solutions to infinite. These solutions were verified and validated using the projective transform and Bertini. Bertini took 20 minutes to find the same set of solutions, the algorithm implemented took more than 3 hours. Bertini find solutions with about 15 significant digits, the algorithm implemented have 12 significant digits. These are the solutions of the polynomial system, now transforming back the variables for the original equations, these values need to be tested if they satisfy the static equilibrium. The condition of static equilibrium used is similar to the first example with the addition of having two points in which the equilibrium can be calculated using the scalar product, see Figure 12 for the free body diagram. It is worth mention that again the

Table 5. Real Solutions Second System

Real Solutions	$x1$	$x2$	$D1$	$D2$	$D3$
1	0	0	-21	3	9
2	0	0	-21	3	-9
3	0	0	-21	-3	9
4	0	0	-21	-3	-9
5	0	0	21	-3	-9
6	0	0	21	-3	9
7	0	0	21	3	-9
8	0	0	21	3	9
9	-0.24945	-2.7095	7.7968	-5.8572	10.932
10	0.24945	2.7095	7.7967	-5.8572	10.932
11	-0.36629	-2.7301	-7.7527	-7.7527	-9.3218
12	0.36629	2.7301	-7.7527	-7.7527	-9.3218
13	-0.36907	-4.0088	-5.8572	7.7968	10.932
14	0.36907	4.0088	-5.8572	7.7968	10.932
15	-0.38544	-1.9356	10.001	-8.0547	-7.0555
16	0.38544	1.9356	10.001	-8.0547	-7.05550
17	-0.38624	-2.589	8.0673	8.0673	8.7688
18	0.38624	2.589	8.0673	8.0673	8.7688
19	-0.50727	-1.9713	9.8698	9.8698	-5.1763
20	0.50727	1.9713	9.8698	9.8698	-5.1763
21	-0.51663	-2.5944	-8.0547	10.001	-7.0555
22	0.51663	2.5944	-8.0547	10.001	-7.0555
23	0.6742	1.4832	12	-12	0
24	0.6742	1.4832	12	12	0
25	0.6742	1.483	-12	12	0
26	0.6742	1.4832	-12	-12	0
27	-0.6742	-1.4832	12	-12	0
28	-0.6742	-1.4832	12	12	0
29	-0.6742	-1.4832	-12	12	0
30	-0.6742	-1.4832	-12	- 12	0

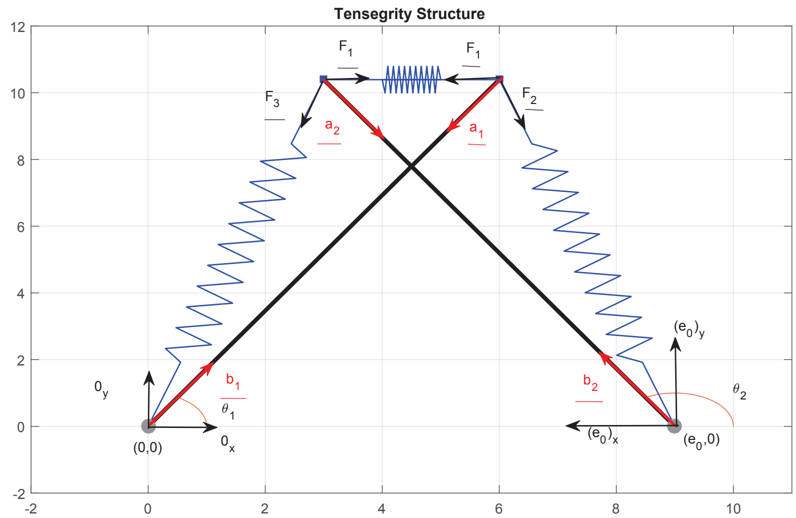

Figure 12. Free Body Diagram Example 2

variables D_1, D_2 and D_3 correspond to the distances between the extremes of the rigid bodies, a distance can not be negative, so this will be to took in to account later. For each par of values of the angles, the scalar products are calculated, that is the way of prove static equilibrium of this tensegrity structure. See Table 6 with the results of the scalar product for each set of angles.

The first 8 solutions have the same value of angles, with different signs in the distances D_1, D_2 and D_3, as in the previous example the negative distances will generate a wrong solution, only one of them have all positive distances but as shown in Figure 13 is a solution with no physical meaning because there are elements that overlap each other, and also one of the scalar products is positive due to this overlap. Solutions 19 to 32, definitely do not fulfill the static equilibrium conditions and they are discarted. Some of them also represent configurations with no physical meaning like the solution 25 shown in Figure 14 when there are two overlaps; between the tensor and the rigid body, and the tensor who is located between the two rigid bodies have a distance of zero, which is not acceptable. Solutions 17 and 18, are valid because comply with the static equilibrium conditions and the distances are positives, the configurations of this two solutions are shown in Figure 15.

Table 6. Comprobation of Static Equilibrium of Real Solutions Second System

Real Solutions	$x1$	$x2$	$theta1$	$theta2$	$ScalarProduct1$	$ScalarProduct2$
1	0	0	0	0	1	-1
2	0	0	0	0	1	-1
3	0	0	0	0	1	-1
4	0	0	0	0	1	-1
5	0	0	0	0	1	-1
6	0	0	0	0	1	-1
7	0	0	0	0	1	-1
8	0	0	0	0	1	-1
9	-0.24945	-2.7095	-0.4889	-2.4345	-0.9574	-1.0000
10	0.24945	2.7095	0.4889	2.4345	-0.9574	-1.0000
11	-0.36629	-2.7301	-0.7022	-2.4394	-0.9990	-0.9990
12	0.36629	2.7301	0.7022	2.4394	-0.9990	-0.9990
13	-0.36907	-4.0088	-0.7071	-2.6527	-1.0000	-0.9574
14	0.36907	4.0088	0.7071	2.6527	-1.0000	-0.9574
15	-0.38544	-1.9356	-0.7358	-2.1879	-0.9988	-0.9631
16	0.38544	1.9356	0.7358	2.1879	-0.9988	-0.9631
17	-0.38624	-2.589	-0.7372	-2.4044	-1.0000	-1.0000
18	0.38624	2.589	0.7372	2.4044	-1.0000	-1.0000
19	-0.50727	-1.9713	-0.9389	-2.2027	-0.9126	-0.9126
20	0.50727	1.9713	0.9389	2.2027	-0.9126	-0.9126
21	-0.51663	-2.5944	-0.9537	-2.4058	-0.9631	-0.9988
22	0.51663	2.5944	0.9537	2.4058	-0.9631	-0.9988
23	0.6742	1.4832	1.1864	1.9552	-0.84120	-0.8238
24	0.6742	1.4832	1.1864	1.9552	-0.84120	-0.8238
25	0.6742	1.4832	1.1864	1.9552	-0.84120	-0.8238
26	0.6742	1.4832	1.1864	1.9552	-0.84120	-0.8238
27	-0.6742	-1.4832	-1.1864	-1.9552	-0.84120	-0.8238
28	-0.6742	-1.4832	-1.1864	-1.9552	-0.84120	-0.8238
29	-0.6742	-1.4832	-1.1864	-1.9552	-0.84120	-0.8238
30	-0.6742	-1.4832	-1.1864	-1.9552	-0.84120	-0.8238

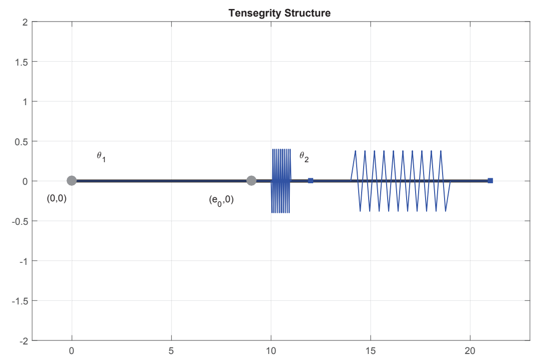

Figure 13. Tensegrity Structure Example 2 Solution 8

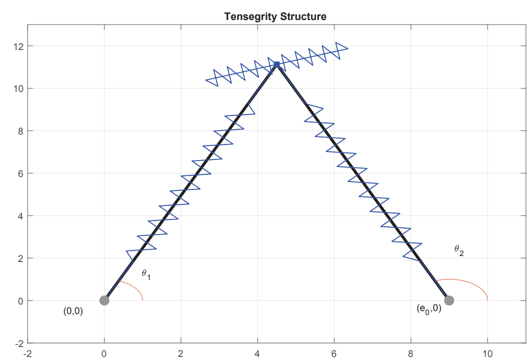

Figure 14. Tensegrity Structure Example 2 Solution 25

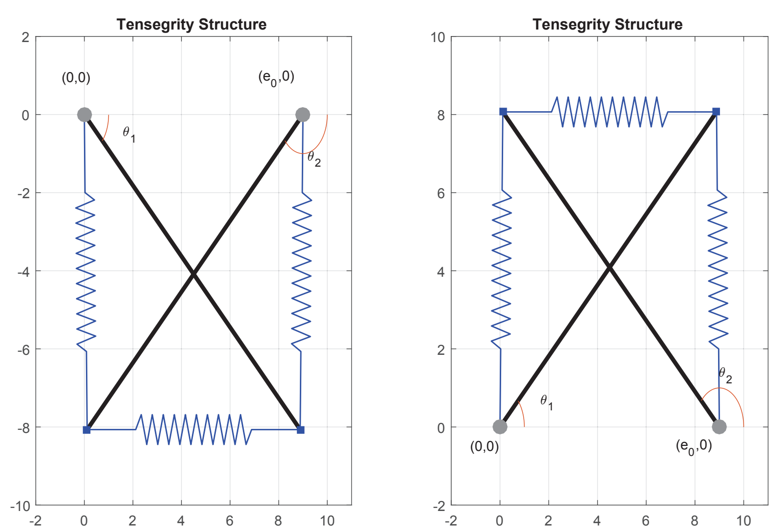

Figure 15. Tensegrity Structure Example 2 Solutions 17 and 18

Solutions 9 to 16 present a patron of scalar products very close to pass the comprobation of the static equilibrium, in some of them one of the scalar product is -1 but no the other, like solutions 9 and 10, but will be treated as wrong solutions. There is also a very close solutions, number 11 and 12, with a scalar product of (-0.9990, -0.9990). Based on the original solution of the polynomial system, more significant digits where included in the solution 12, looking to find a good solution if the resolutions where better, the values x_1=0.366289252845424, x_2=2.730083098597832, where tested and the results of the scalar products stay the same (-0.9990, -0.9990) with the configuration shown in the Figure 16, these two solutions do not fit enough to be treated as good solutions.

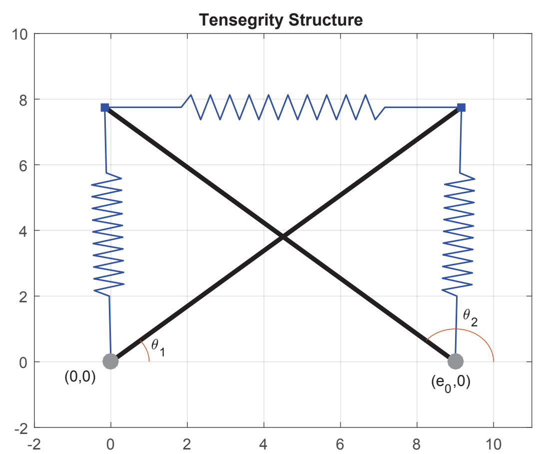

Figure 16. Tensegrity Structure Example 2 Solution 12

5.4. GRAPHICAL INTERFACE

As mentioned the continuation method was implemented in MATLAB for each tensegrity structure, the code and the graphical interface is included in the work, this will allow the user the verification of the results. In Figure 17 the interface is shown with the parameters that the user can change and the visualization for the two structures analyzed in this process. A manual of how to use the interface is found in the appendices.

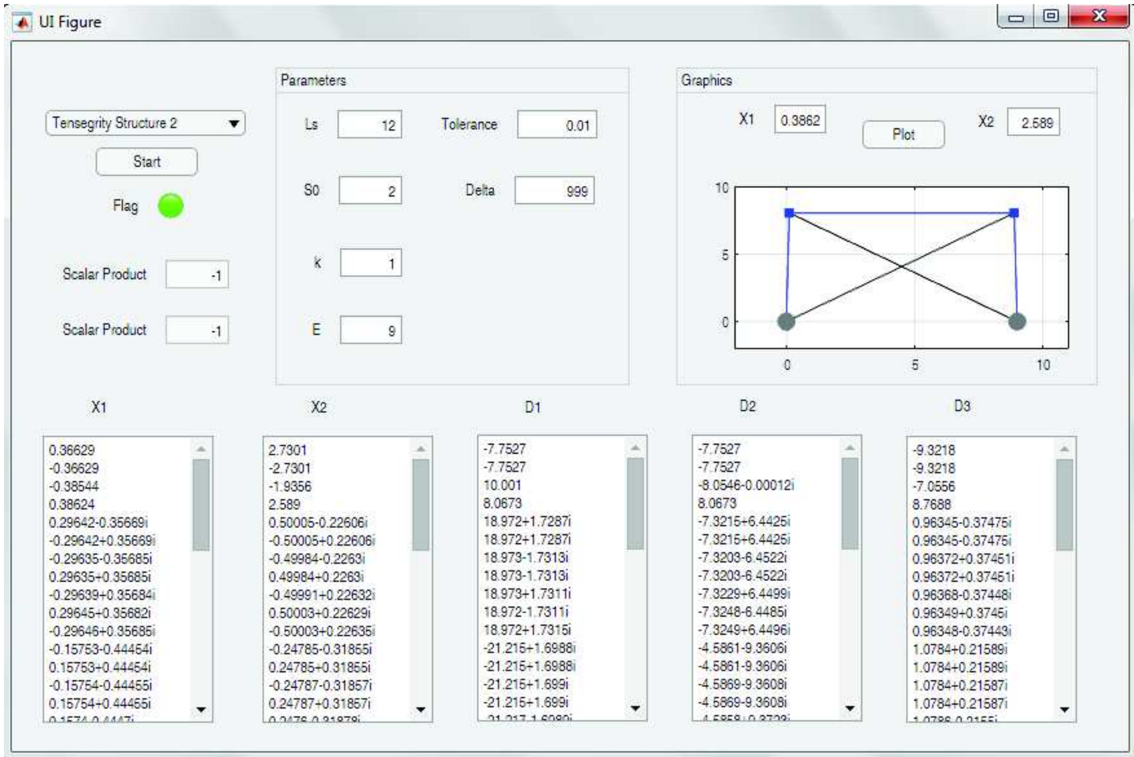

Figure 17. Graphical Interface

6. CONCLUSIONS AND FUTURE WORK

6.1. CONCLUSIONS

Two tensegrity structures were analyzed for static equilibrium, looking for different configurations, the analysis was accomplished using the virtual work principle. This approach was used once in the past, but then the process and the generated polynomial system through the addition of new variables is different to the one used here, the objective was looking to find more configurations using the continuation method. At the end the polynomial system yield a set of possible real solutions, these solutions where tested to fulfill the static equilibrium condition, finding two solutions for each tensegrity structure, this results are consistent with other numeric methods like Newton-Raphson but with the difference that no initial condition specific to the problem had to be given for the method to work.

In the algorithm implementation, the chosen starting system provided a set of solutions that allowed to start each path in the continuation method. This starting system was easy to solve and to scale, from the first examples to the final algorithm. The homotopy function implemented allowed to link the starting system with the original system. This function was easy to implement and allowed to obtain all the possible solutions of the polynomial systems. The parameter t, allowed to avoid crossing paths and to found all possible solutions. In this implementation the value of t took very small fixed steps, as a consequence the algorithm have great computational cost.

For each tensegrity structure an algorithm of the continuation method was implemented, the solution of each polynomial system is tested by two methods, first the projective transform which was also implemented, and second using Bertini, a software implemented by others authors, in both cases the solutions were the same, validating the results.

For the comprobation of static equilibrium state, the analysis used was: The scalar product of the unit vector of the summation of forces in a strut, with the unit vector that specifies the direction of the strut should be -1. For each tensegrity structure two con?gurations were found that accomplished this condition.

The continuation method algorithm implemented, does not have the best performance in time and accuracy in comparison with Bertini, but the code of Bertini is not accessible only the execution is available. This implementation could be used as teaching tool of the continuation method concepts this also could be taken in to account when someone wants to using the method for future work.

6.2. FUTURE WORK

As for the static equilibrium analysis for tensegrity structures, the next step could be an asymmetrical configuration of the planar structure or in a mayor scale a three dimensional structure, in which the addition of a new dimension could be the key for finding new configurations. This situation was explored in this work, a polynomial system was created for a 3d prism tensegrity structure, following the same methodology, the polynomial system generated had a degree of millions, becoming a very complex problem itself, but given the possibility that new configuration could be found and this is possible through the continuation method. It is something that will be looked for, the advise for this future work are, in the process of generating the the polynomial system, an approach of reducing the number of equations and variables, as the degree of the polynomial system should be the main focus of the problem. This is an intuitive process there are not special steps or one algorithm that can make this analysis.

The continuation method implemented in this work was very straightforward specially in the choice of the parameter t but this produce great computational cost. For future works more robust path tracking algorithms should be implemented, to lower the computational time and still find all the possible solutions of the polynomial system. Another improvement could be a parallel implementation, the method is suited for this programming technique because each path can be track independent of other paths.

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APPENDICES

A. GRAPHICAL INTERFACE MANUAL

A.1. TECHNICAL REQUIREMENTS

This interface can be found as a MATLAB app (*TensegrityStructureAnalysis.mlappinstall*), which can be installed in versions of MATLAB 2016a and forward. The source code for open and edit through the app designer (*InterfazGrafica.mlapp*) is also shared. If this interface is open, there could be an overlap of some of the labels depending of the screen resolution (1366x768 is recommended), but it will not affect the results of the analysis of the structure. Also the code of each implementation can be run in MATLAB separated from the user interface, this code is more suitable for changes, specially if others structures want to be implemented, this code does require the Symbolic Math Toolbox, for the generation of the starting system. The code is labeled as (*CONSOLxx.m*) where *xx* are different implementation for the continuation method for the polynomial systems solved in this work.

A.2. FIRST TENSEGRITY STRUCTURE

The graphical interface build is very simple to use and is based on the implementation of the two tensegrity structures stated in this work. The first step is choosing which tensegrity structure is going to be solved, in this case tensegrity structure 1, is a drop down menu with only two options, in Figure 18 is shown, also at the bottom of drop down menu, is located the start button, which will start the algorithm and solve the polynomial system based in the tensegrity structure and the other parameters chosen. Also an indicator flag will show a green light when the polynomial finished the execution of the algorithm.

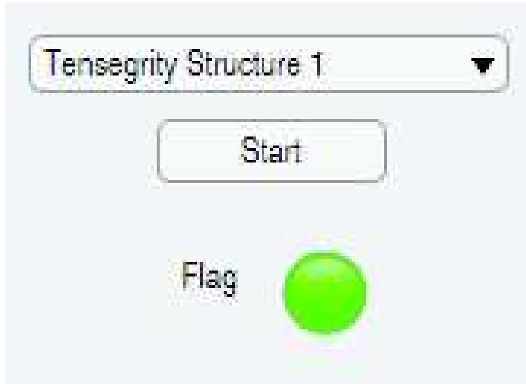

Figure 18. Graphical Interface Drop Down First Tensegrity Structure

Then a set of parameters can be chosen based on the tensegrity structure, for the first tensegrity structure we have:

- *Ls*, is the length of the rigid body, the values should be positive numbers. The value used was 12.
- S_0 is the free length of the tensors or springs, is the same for the two of them,the values should be positive numbers. The value used was 2.
- k is the stiffness constant of the tensors or springs, the values should be positive numbers. The value used was 1.
- a is the location of the first tensor or spring on the x axis, for this structure the value on the y axis is 0. The value used was 16
- b is the location of the second tensor or spring on the y axis, for this structure the value on the x axis is 0. The value used was 10
- $Tolerance$ is the tolerance in the newton method of each iteration, if the value is low, more resolution for the solutions will be obtained, but the computational cost can be high. The value used was 0.001
- $Delta$ is the number of steps the continuation method will follow from 0 to 1, between more step better resolutions and guarantee find all the solutions but it will be a more intensive computation. Value used was 99999.

This values are located in the panel of the middle see Figure 19. This interface is not built as a generic solver, the purpose is to work for the tensegrity structures implemented, and for parameters close to the ones evaluated. If parameters are to far away in comparison with the ones implemented, the solutions are no guaranteed.

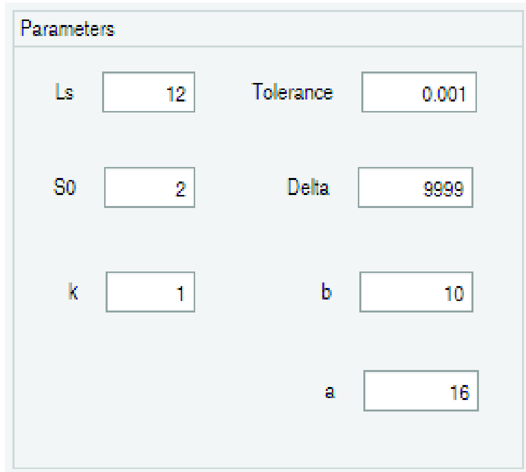

Figure 19. Graphical Interface Panel First Tensegrity Structure

With the values entered correctly, click the start button, after the method finish it will print the solutions in the three boxes in the bottom of the interface see Figure 20

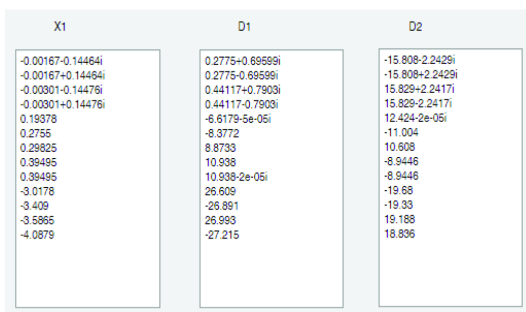

Figure 20. Graphical Interface Solutions First Tensegrity Structure

After getting the results, one of the real results of the first column $X1$ must be chosen to generate the visualization of the configuration of the tensegrity structure, this should be entered manually in the graphics panel, variable $X1$, and click in the button *Plot*, This will show the Figure 21 of the configuration, the blue lines are the tensors or springs, the black line is the rigid body, the circle marker is a pivot that depends of the parameters of the structure, and the square marker is given by the angle calculated from the solution.

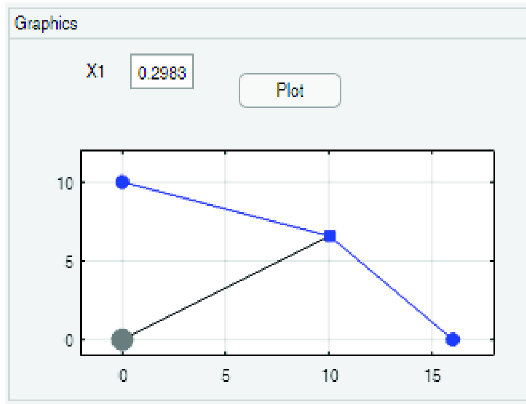

Figure 21. Graphical Interface Configuration First Tensegrity Structure

And also will show the result of the scalar product this should be -1 for validate the solution, see Figure 22

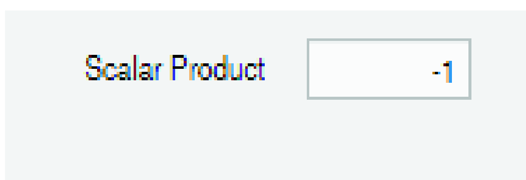

Figure 22. Graphical Interface Scalar Product First Tensegrity Structure

A.3. SECOND TENSEGRITY STRUCTURE

Tensegrity Structure 2, is chosen from the drop down menu, in Figure 23 is shown, also at the bottom of drop down menu, is located the start button, which will start the algorithm and solve the polynomial system based in the tensegrity structure and the other parameters chosen. Also an indicator flag will show a green light when the execution of the algorithm.

Figure 23. Graphical Interface Drop Down Second Tensegrity Struture

Then a set of parameters can be chosen based on the tensegrity structure, for the second tensegrity structure we have:

- *Ls* is the length of the rigid bodies, the values should be positive numbers. The value used was 12.
- S_0 is the free length of the tensors or springs, is the same for the two of them,the values should be positive numbers. The value used was 2.
- k is the stiffness constant of the tensors or springs, the values should be positive numbers. The value used was 1.
- E is the location of the pivot of the second rigid body on the x axis, for this pivot the value on the y axis is 0. The value used was 9
- $Tolerance$ is the tolerance in the newton method of each iteration, if is a low value, more resolution for the solutions will be obtained, but the computational cost can be high. The value used was 0.00001
- $Delta$ is the number of steps the continuation method will follow from 0 to 1, between more step better resolutions and guarantee find all the solutions but it will be a more intensive computation. Value used was 99999.

This values are located in the panel of the middle see Figure 24. This interface is not a built as a generic solver, is meant to work for the tensegrity structures implemented, and for parameters close to the ones evaluated, if parameters are to far away in comparison with the ones implemented, the solutions are no guaranteed.

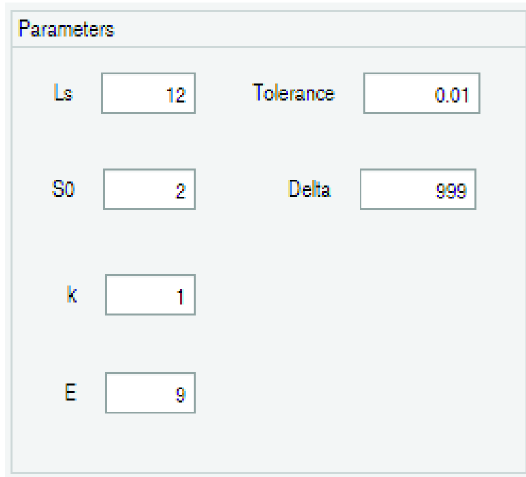

Figure 24. Graphical Interface Panel First Tensegrity Structure

With the values entered correctly, click the start button, after the method finish it will print the solutions in the five boxes in the bottom of the interface see Figure 27

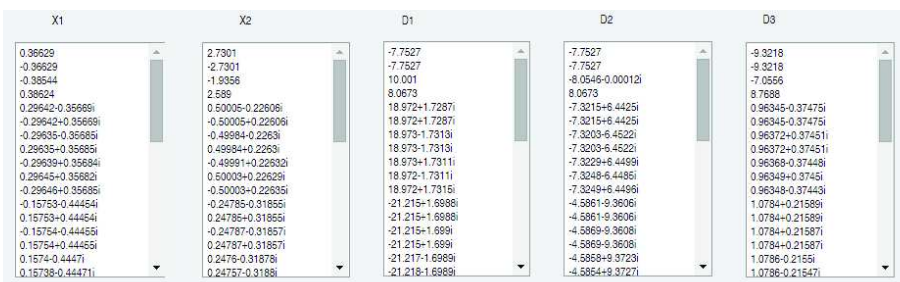

Figure 25. Graphical Interface Solutions First Tensegrity Structure

After getting the results, the solutions from the first and second column ($X1$ and $X2$) must be chosen to generate the visualization of the configuration of the tensegrity structure (the par values chosen have to be exactly in the same row,), this should be entered manually in the Graphics panel, in variable $X1$ and variable $X2$, and click in the button *Plot*, This will show the Figure 26 of the configuration, the blue lines are the tensors or springs, the black line is the rigid body, the gray circle markers are the pivot of the rigid bodies, and the square marker is the union between tensors and rigid bodies given by the angles calculated from the solutions.

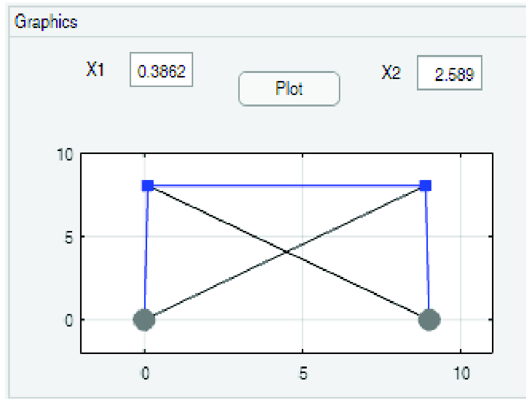

Figure 26. Graphical Interface Configuration First Tensegrity Structure

And also will show the result of the scalar products this should be -1 in both calculations for validate the solution, see Figure 27

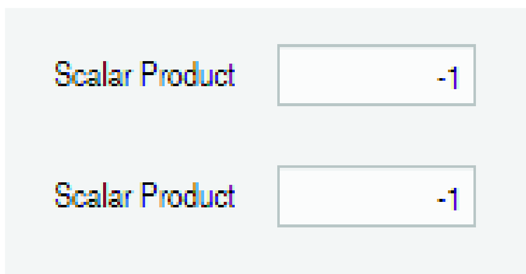

Figure 27. Graphical Interface Scalar Product First Tensegrity Structure

B. PAPER

Solving a Static Equilibrium Problem of a Tensegrity Structure Using the Continuation Method

José D. Montoya[*] Julio C. Correa[†]

November 23, 2016

Abstract

This work addresses the static equilibrium analysis of a tensegrity structure. This analysis is perform using the virtual work principle obtaining a system of equilibrium equations. This equations are transformed to a polynomial system. Then an implementation of the continuation method will be used for find all possible solutions to the polynomial system. The real solutions obtained by the continuation method will be tested to fulfill the conditions of static equilibrium. The objective is to find different sets of solutions that might lead to new configurations of the tensegrity structures.

Keywords: Continuation method, tensegrity structure, polynomial system, static equilibrium

1 Introduction

In engineering, tensegrity structures are used in several fields like, architecture, energy generation and aerospace. These structures have two principal components, rigid bodies or compressive parts and strings or tensile parts. A challenge that the engineers face when working with these structures, is the static equilibrium analysis, the tensegrity structures need to have a balance of forces to obtain a state of static equilibrium [13]. For solving this problems, the use of numerical methods is important to find the correct design of the tensegrity structures, when the static analysis is perform usually a lot of equations arise, solutions of the numerical methods currently employed are usually local solutions starting with a initial condition, the though of having a numerical method that will solve the static analysis of a tensegrity structure and obtain all the possible solutions, could give a lot of new configurations for the static equilibrium problem and could be very useful to the engineers working with the tensegrity structures. Tensegrity structures have been studied and applied in a several fields, like civil engineering [4] , aerospace [3], wave energy harvesting [12], human anatomy [10], modeling the motion viruses [11] and aquaculture [5]. The analysis of tensegrity structures with numerical methods is in continuous research, specially for robotics applications as noted for McCarthy [8]. Bayat and Crane [2] also have try to get more solutions for the static analysis of tensegrity structures. Jiang and Vijay [6] have conducted similar analysis of objects suspended from cables using the continuation method. The objective of this work is to find different sets of solutions that might lead to new configurations of the tensegrity structure analyzed.

2 Problem definition

Figure 1 shows the tensegrity structure chosen. This tensegrity structure consist of two rigid bodies, and three springs connecting each strut.

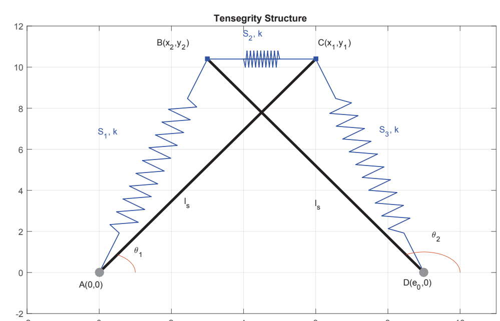

Figure 1: Tensegrity Structure

- The origin, is a fixed pivot in (0,0) for the first rigid body and a fixed pivot in (e_0,0) for the second rigid body
- The location of the tensor S_1 is between the fixed pivot (0,0), and the free end of the second rigid Body
- The location of the tensor S_2 is between the the free end point of the two rigid bodies
- The location of the third tensor S_1 is between the fixed pivot (e_0,0), and the free end of the first rigid body
- The length of the rigid bodies is l_s
- The angle θ_1 is measured for the first rigid body
- The angle θ_2 is measured for the second rigid body
- The length of each spring is equal S_0
- The stiffness of each spring is equal to k

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3 Static Equilibrium Analysis

The potential energy stored in each spring is given by

$$V_i = \frac{1}{2} k_i (S_i - S_{0i})^2 \tag{1}$$

Where S_i is the actual length of the spring. For Example, for the third spring.

$$S_3 = \sqrt{(C_x - e_0)^2 + C_y^2} \tag{2}$$

Where $C_x = l_s Cos\theta_1$ and $C_y = l_s Sin\theta_1$
When the contribution to the potential energy provided by both springs are annulled, the resultant expression is function of θ. This is

$$V(\theta) = V_1 + V_2 \tag{3}$$

The principle of virtual work establishes that

$$\frac{\partial V}{\partial \theta} = 0 \tag{4}$$

After following the procedure described here for all the components, this equations are obtained:

$$F(1) = l_s^2 cos(\theta_1) sin(\theta_1) - l_s^2 cos(\theta_1) sin(\theta_2) + 2e_0 l_s sin(\theta_1) - \frac{e_0 l_s s_0 sin(\theta_1)}{\sqrt{e_0^2 - 2e_0 l_s cos(\theta_1) + l_s^2}} + \frac{l_s^2 s_0 cos(\theta_1) sin(\theta_2) - l_s^2 s_0 cos(\theta_2) sin(\theta_1) - e_0 l_0 s_0 sin(\theta_1)}{\sqrt{Aux_1}} \tag{5}$$

$$F(2) = l_s^2 cos(\theta_1) sin(\theta_2) - l_s^2 cos(\theta_2) sin(\theta_1) - 2e_0 l_s sin(\theta_2) + \frac{e_0 l_s s_0 sin(\theta_2)}{\sqrt{e_0^2 + 2cos(\theta_2) e_0 l_s + l_s^2}} + \frac{-(l_s^2 s_0 cos(\theta_1) sin(\theta_2)) + l_s^2 s_0 cos(\theta_2) sin(\theta_1) + e_0 l_s s_0 sin(\theta_2)}{\sqrt{Aux_1}} \tag{6}$$

where
$Aux_1 = e_0^2 - 2e_0 l_s cos(\theta_1) + 2e_0 l_s cos(\theta_2) + l_s^2 cos(\theta_1)^2 - 2l_s^2 cos(\theta_1) cos(\theta_2) + l_s^2 cos(\theta_2)^2 + l_s^2 sin(\theta_1)^2 - 2l_s^2 sin(\theta_1) sin(\theta_2) + l_s^2 sin(\theta_2)^2$

4 Continuation Method

Li [7], explains how the continuation method works. The idea for solving a polynomial systems is to start with a simpler polynomial system that is easy to solve and to obtain all the solutions, and with the aid of a function, link it with the wanted polynomial system. This is done through a parameter *t*, which range is between 0 and 1, when the value of *t=0*, the polynomial system to solve is the simpler one, when the value of *t=1* the polynomial system to solve its the desired one usually more complex.

The two polynomial systems linked by the homotopy function will look like this:

$$h_j(x,t) = (1-t) g_j(x) + t f_j(x) \tag{7}$$

Where j=1,2,..., n. d_j is the degree of the polynomial, g_j is the starting system and f_j is the original polynomial system, t as stated before is the variable that will change from 0 to 1 each step. The starting system also have to be chosen to be very simple and easy to scale when higher degree system start to arise, the structure of the starting system used is the one used in Morgan [9]:

$$g_j(x) = p_j^{d_j} x_j^{d_j} - q_j^{d_j} \tag{8}$$

Where j=1,2,..., n. d_j is the degree of the polynomial, p_j and q_j are the complex constant. x_j is the variable of the polynomial system.
The starting system must have the same degree of the original, and the solutions must have a certain characteristic: independence, which is obtained by selecting random constants as coefficients, Morgan [9] explains that this independence will allow to find all the solutions of the original polynomial system. The continuation method can be used directly with the polynomial system, but sometimes is not enough, because some of the solutions can be missed, some path could cross or be mistaken by a solution to infinite. The way to find if a solution is at infinite is using the projective transformation [9].This transformation will not be explain here, but for general knowledge this transformation can be associated with a change of base in linear algebra, as a result another polynomial system will be generated.

This continuation method can only be applied to polynomial system, and the equations derived from the virtual work principle have denominators, trigonometric functions, and square roots. With a variable change a polynomial system can be generated, as a results five polynomials with five variables where obtained: D_1, D_2, D_3,

X_1 and X_2

$$Poly1 = 2D_2D_3l_s^2k_0X_1 - 2D_2D_3l_s^2k_0X_2 - 2D_2l_s^2s_0k_0X_1 + 2D_2l_s^2s_0k_0X_2 - 2D_2D_3l_s^2k_0X_1X_2^2 + 2D_2D_3l_s^2k_0X_1^2X_2 + 4D_2D_3e_0l_sk_0X_1 + 2D_2l_s^2s_0k_0X_1X_2^2 - 2D_2l_s^2s_0k_0X_1^2X_2 - 2D_2e_0l_ss_0k_0X_1 - 2D_3e_0l_ss_0k_0X_1 + 4D_2D_3e_0l_sk_0X_1X_2^2 - 2D_2e_0l_ss_0k_0X_1X_2^2 - 2D_3e_0l_ss_0k_0X_1X_2^2 \quad (9)$$

$$Poly2 = 2D_1D_3l_s^2k_0X_2 - 2D_1D_3l_s^2k_0X_1 + 2D_1l_s^2s_0k_0X_1 - 2D_1l_s^2s_0k_0X_2 + 2D_1D_3l_s^2k_0X_1X_2^2 - 2D_1D_3l_s^2k_0X_1^2X_2 - 4D_1D_3e_0l_sk_0X_2 - 2D_1l_s^2s_0k_0X_1X_2^2 + 2D_1l_s^2s_0k_0X_1^2X_2 + 2D_1e_0l_ss_0k_0X_2 + 2D_3e_0l_ss_0k_0X_2 - 4D_1D_3e_0l_sk_0X_1^2X_2 + 2D_1e_0l_ss_0k_0X_1^2X_2 + 2D_3e_0l_ss_0k_0X_1^2X_2 \quad (10)$$

$$Poly3 = D_1^2X_2^2 + D_1^2 - e_0^2X_2^2 - e_0^2 + 2e_0l_sX_2^2 - 2e_0l_s - l_s^2X_2^2 - l_s^2 \quad (11)$$

$$Poly4 = D_2^2X_1^2 + D_2^2 - e_0^2X_1^2 - e_0^2 - 2e_0l_sX_1^2 + 2e_0l_s - l_s^2X_1^2 - l_s^2 \quad (12)$$

$$Poly5 = D_3^2X_1^2X_2^2 + D_3^2X_1^2 + D_3^2X_2^2 + D_3^2 - e_0^2X_1^2X_2^2 - e_0^2X_1^2 - e_0^2X_2^2 - e_0^2 - 4e_0l_sX_1^2 + 4e_0l_sX_2^2 - 4l_s^2X_1^2 + 8l_s^2X_1X_2 - 4l_s^2X_2^2 \quad (13)$$

5 Results

This polynomial system was implemented using the continuation method, with the goal of finding all the solutions that satisfy the polynomial system, the original system and have physical meaning. A set of constants with a predefined value where chosen: e_0=9, l_s=12, s_0=2, k_0=1, to implement this polynomial system numerically.

This polynomial system have degree of 2400, which mean 2400 paths have to be tracked and find equal number of solutions, after the implementation 108 possible solutions where found, 30 of them reals and are shown in Table 1, the rest of the solutions are solutions to infinite.

With these solutions we have to obtain the original unknowns θ_1 and θ_2 transforming back the variables. Real values need to be checked if they satisfy the static equilibrium condition. Since the strut is a member under two forces, one at the fixed and the other at the free end, both forces must be equal in magnitude but with opposite directions. In other words, the unit vector of the spring forces acting on the free end of the strut must be in the opposite direction of the unit vector that goes form the fixed end to the free end. Therefore the scalar product between these unit vectors must be -1 in the equilibrium, see Figure 2 for the free body diagram. For this calculation the only variables needed are the angles. It is worth to mention that since variables D_1, D_2 and D_3 are distances they can not be negative, This will be to took in to account later.

For each par of values of the angles, the scalar products are calculated, that is the way of prove static equilibrium of this tensegrity structure. See Table 2 with the results of the scalar product for each set of angles.

Table 1: Real Solutions Second System

Solutions	$x1$	$x2$	$D1$	$D2$	$D3$
1	0	0	-21	3	9
2	0	0	-21	3	-9
3	0	0	-21	-3	9
4	0	0	-21	-3	-9
5	0	0	21	-3	-9
6	0	0	21	-3	9
7	0	0	21	3	-9
8	0	0	21	3	9
9	-0.24945	-2.7095	7.7968	-5.8572	10.932
10	0.24945	2.7095	7.7967	-5.8572	10.932
11	-0.36629	-2.7301	-7.7527	-7.7527	-9.3218
12	0.36629	2.7301	-7.7527	-7.7527	-9.3218
13	-0.36907	-4.0088	-5.8572	7.7968	10.932
14	0.36907	4.0088	-5.8572	7.7968	10.932
15	-0.38544	-1.9356	10.001	-8.0547	-7.0555
16	0.38544	1.9356	10.001	-8.0547	-7.05550
17	-0.38624	-2.589	8.0673	8.0673	8.7688
18	0.38624	2.589	8.0673	8.0673	8.7688
19	-0.50727	-1.9713	9.8698	9.8698	-5.1763
20	0.50727	1.9713	9.8698	9.8698	-5.1763
21	-0.51663	-2.5944	-8.0547	10.001	-7.0555
22	0.51663	2.5944	-8.0547	10.001	-7.0555
23	0.6742	1.4832	12	-12	0
24	0.6742	1.4832	12	12	0
25	0.6742	1.483	-12	12	0
26	0.6742	1.4832	-12	-12	0
27	-0.6742	-1.4832	12	-12	0
28	-0.6742	-1.4832	12	12	0
29	-0.6742	-1.4832	-12	12	0
30	-0.6742	-1.4832	-12	- 12	0

The first 8 solutions have the same value of angles, with different signs in the distances D_1, D_2 and D_3, as in the previous example the negative distances will generate a wrong solution, only one of them have all positive distances but as shown in Figure 3 is a solution with no physical meaning because there are elements that overlap each other, and also one of the scalar products is positive due to this overlap.

Solutions 19 to 32, definitely do not fulfill the static equilibrium conditions and they are discarted. Some of them also represent configurations with no physical meaning like the solution 25 shown in Figure 4 when there are two overlaps; between the tensor and the rigid body, and the tensor who is located between the two rigid bodies have a distance of zero, which is not acceptable.

Solutions 9 to 16 present a patron of scalar products very close to pass the comprobation of the static equilibrium, in some of them one of the scalar product is -1 but no the other, like solutions 9 and 10, but will be treated as wrong solutions. There is also a very close solutions, number 11 and 12, with a scalar product of (-0.9990, -0.9990), based on the original solution of the polynomial system, more significant digits where included in the solution

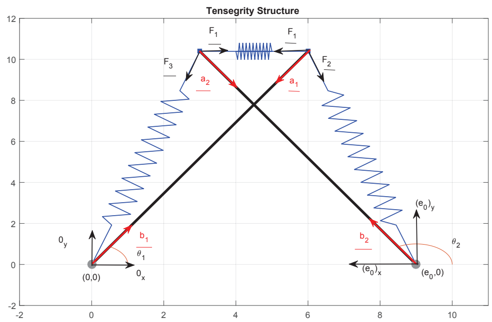

Figure 2: Free Body Diagram

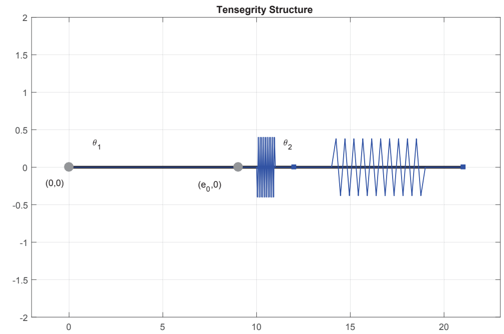

Figure 3: Tensegrity Structure Solution 8

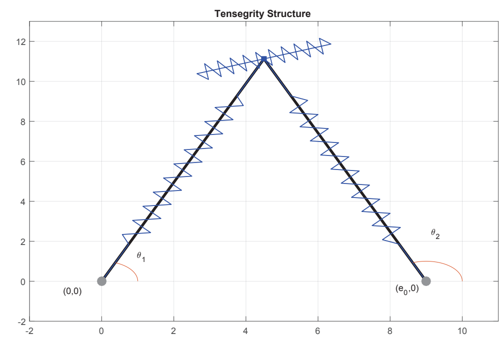

Figure 4: Tensegrity Structure Solution 25

Table 2: Comprobation of Static Equilibrium of Real Solutions Second System

Real Solutions	θ_1	θ_2	*Scalar Product*1	*Scalar Product*2
1	0	0	1	-1
2	0	0	1	-1
3	0	0	1	-1
4	0	0	1	-1
5	0	0	1	-1
6	0	0	1	-1
7	0	0	1	-1
8	0	0	1	-1
9	-0.4889	-2.4345	-0.9574	-1.0000
10	0.4889	2.4345	-0.9574	-1.0000
11	-0.7022	-2.4394	-0.9990	-0.9990
12	0.7022	2.4394	-0.9990	-0.9990
13	-0.7071	-2.6527	-1.0000	-0.9574
14	0.7071	2.6527	-1.0000	-0.9574
15	-0.7358	-2.1879	-0.9988	-0.9631
16	0.7358	2.1879	-0.9988	-0.9631
17	-0.7372	-2.4044	-1.0000	-1.0000
18	0.7372	2.4044	-1.0000	-1.0000
19	-0.9389	-2.2027	-0.9126	-0.9126
20	0.9389	2.2027	-0.9126	-0.9126
21	-0.9537	-2.4058	-0.9631	-0.9988
22	0.9537	2.4058	-0.9631	-0.9988
23	1.1864	1.9552	-0.84120	-0.8238
24	1.1864	1.9552	-0.84120	-0.8238
25	1.1864	1.9552	-0.84120	-0.8238
26	1.1864	1.9552	-0.84120	-0.8238
27	-1.1864	-1.9552	-0.84120	-0.8238
28	-1.1864	-1.9552	-0.84120	-0.8238
29	-1.1864	-1.9552	-0.84120	-0.8238
30	-1.1864	-1.9552	-0.84120	-0.8238

12, looking to find a good solution if the resolutions where better, the values x_1=0.366289252845424, x_2=2.730083098597832, where tested and the results of the scalar products stay the same (-0.9990, -0.9990) with the configuration shown in the Figure 5, these two solutions do not fit enough to be treated as good solutions.

Solutions 17 and 18, are valid because comply with the static equilibrium conditions and the distances are positives, the configurations of this two solutions are shown in Figure 6.

6 Conclusions

A tensegrity structure was analyzed for static equilibrium, looking for different configurations, the analysis was perform using the virtual work principle. This approach was used once in the past, but then the process and the generated polynomial system through the addition of new variables is different to the one used here, the objective was looking to find more configurations using the continuation method. At the end the polynomial system yield a set of possible real solutions, these solutions where tested to fulfill the static equilibrium condition, finding two solutions, this results are consistent with other numeric methods like Newton-Raphson but with the difference that no initial condition specific to the problem had to be given for the method to work.

For the tensegrity structure an algorithm of the continuation method was implemented, the solution of each polynomial system is tested by two methods, first the projective transform which was also implemented, and second using Bertini[1] , a software implemented by others authors, in both cases the solutions were the same, validating the results.

The method of comprobation of static equilibrium was analyzed in each rigid body of the tensegrity structure, using the scalar pro-

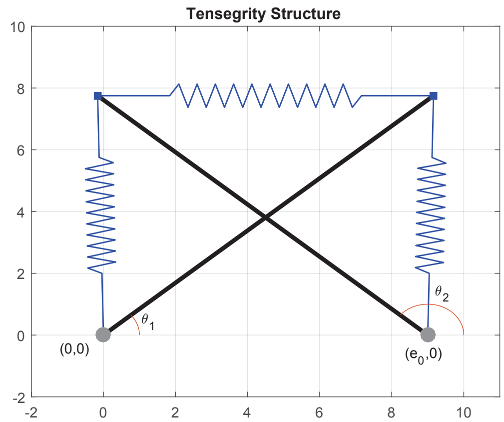

Figure 5: Tensegrity Structure Solution 12

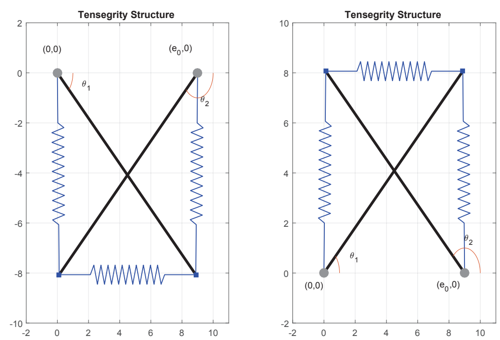

Figure 6: Tensegrity Structure Solutions 17 and 18

duct of the unit vector of the summation of forces in a strut with the unit vector that specifies the direction of the strut, looking for a results of -1 to achieve static equilibrium, for the tensegrity structure two configurations were found

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C. PROPOSAL

Algorithm Implementation of the Continuation Method for Solving a Static Equilibrium Problem of a Tensegrity Structure

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Propuesta de trabajo de grado para optar al título de Magíster en Ingeniería

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ABSTRACT

Problems in mechanisms analysis and robotics lead naturally to polynomial systems, there are methods that solve this problems, like the Newton method, but in a lot of cases not all the solutions are found or only local ones are. One of the analysis that could lead to have complex polynomial systems is the static equilibrium analysis of tensegrity structures, which are structures used in fields like architecture, engineering and biomedicine. When doing this analysis we would like to know all the possible configuration of the elements of a tensegrity structure in which its static equilibrium state its achieved, in this situation is when the continuation method is useful, because is an exhaustive method that find all the solutions. A static equilibrium problem for a tensegrity structure will be studied, obtaining a polynomial system, that will be solved using an algorithm implementation of the continuation method, finding all the possible solutions.

KEYWORDS:

Continuation method, tensegrity structures, polynomial system, static equilibrium.

1. INTRODUCTION AND PROBLEM DEFINITION

In engineering polynomial systems arise naturally and the use and implementation of numerical methods to solve this systems have been very useful, there are specific problems that require for example solve large polynomial systems and obtain all the possible solutions and this specific approach is still a subject of research.

For solving polynomial system there are different methods; symbolic methods, numerical methods and geometric methods. Numerical methods become important when the polynomial system is non trivial, have large degree with many equations and variables, the symbolic methods for example, from 25th degree and forward, could start to generate not so good results in the computational implementation [17].

Numerical methods used for solving polynomial system are classified in iterative methods, like newton's Method, that are good to find local solutions and require having a good starting condition; and continuation methods, that are appropriate for find all the solutions, they create paths that will lead to obtain all the possible solutions, but require more computational effort [17].

Polynomial systems arise naturally in problems like formula construction, geometric intersection problems, inverse kinematics, power flow problems with PQ-specified bases (a problem in which the real and reactive power are speci?ed, and the voltage will be calculated), computation of equilibrium states, etc. For solving this systems we are concerned with the robustness of the methods used and want to be sure that all isolated solutions are obtained, we want exhaustive methods, the continuation method answer all this characteristics [15].

The continuation method consist in start solving a simpler polynomial system and follow a path of solutions that will lead you to one polynomial system and its solutions. The function that link the simpler polynomial system and the complex and wanted polynomial system its called homotopy function.

In engineering, tensegrity structures are used in several fields like, architecture, energy generation and aerospace. This structures have two principal components a rigid bodies or compressive parts and strings or tensile parts, one of the problems engineers face when working with this structures, are the initial configurations, which are very important because they need to have a balance of forces to obtain it static equilibrium state [37], for solving this problems, the use of numerical methods its important for the correct design of the tensegrity structures, when the static analysis its done usually a lot of equations arise, solutions of the numerical methods currently employed are usually local solutions starting with a initial condition, the though of having a numerical method that will solve the static analysis of a tensegrity structure and obtain all the possible solutions, could give a lot of new configurations for its static equilibrium problem and could be very useful to the engineers working with the tensegrity structures

2. STATE OF ART

2.1. CONTINUATION METHOD

The continuation method was first introduced in the 80s, its a method that can find all solutions of polynomial systems with efficiency and reliability; the first applications used with the continuation method was solving kinematics problem [18]. The only limitation of the method is in the computational implementation, it could require intensive computation. During two decades Sommense, Wampler, Morgan, Allgower and Li, did an exhaustive study of the method [20, 21, 22, 14, 2, 27]. The method was applied in different areas with special focus in robotics, in 1985 Tsai y Morgan [30] implemented an algorithm SYMPLE using the continuation method for solving the kinematics of the most general six and five degree of freedom manipulators by continuation methods. Su [28] did an inverse static analysis of planar compliant mechanisms, Lee [12] used it for solving the geometric design problem of spatial 3R robot manipulators using polynomial homotopy continuation [12], Bin [3] used the Continuation method applied in kinematics of parallel robot, for more examples see [29, 31, 35, 16, 19].

There are some software implementation of the continuation method in different programming languages like FORTRAN, C and MATLAB for solving polynomial systems. Verschelde implemented an Algorithm of general purpose PHPack [33, 34], Lee et al implemented the HOM4ps2.0 [13], with an update HOM4ps3.0 [5] that use parallel implementation, Zang implemented a numeric toolbox NACLab [36] for MATLAB that include functionality to work with continuation methods. HOMOPACK developed by Irani et al [10] and HOMOLAB [1] implemented in MATLAB.

Li [14], explain how the continuation method works, the idea for solving a polynomial systems its start with a simpler polynomial system that is easy to solve and obtain all the solutions, and through a function relate it with the polynomial system that we want to solve, this is done through a parameter t, which range is between 0 and 1, when the value of $t=0$, the polynomial system to solve is the simpler one, when the value of *t=1* the polynomial system to solve its the desired one usually more complex, we shown a specific case: We have a polynomial system we want to solve

$$x^2 + y^2 = 5 \tag{1}$$

$$x - y = 1 \tag{2}$$

We start solving a simpler system,

$$x^2 = 1 \tag{3}$$

$$y = 1 \tag{4}$$

This system is easy to solve through inspection. Now we introduce the parameter *t* and link the systems in a function with the parameter *t*,

$$(1-t)\begin{bmatrix} x^2 - 1 \\ y - 1 \end{bmatrix} + t\begin{bmatrix} x^2 + y^2 - 5 \\ x - y - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

when in the system (5) *t=0*, we obtain the system (1)(2) and when *t=1* we have the system (3)(4), this type of functions are known as homotopy functions. The homotopy functions could have different structures or modifications according to the application or the implementation of the continuation method.

Morgan explains in his book [20], two important characteristics of the continuation method: for each displacement of the *t* the method use the previous solution of the intermediate polynomial system and apply the newton method, smaller displacements of *t* more intermediate system will be solved, there will be more computational demand, but in some cases this will avoid the cross of the solutions paths. Another important characteristic is, what its obtained in the continuation method are paths for each solutions that, starts in *t=0* and end in *t=1*; in the end it will be easy to tell which solutions are general, singular, infinite solutions or solutions to infinite, the visual representation its very helpful to discover the solutions that exists at the end of the path which are the solution that have physical meaning. In figure 1 there are two paths, each solution generate a path though the t axis with real and imaginary part. Also in figure 2 the projection of the solution its shown in the t axis

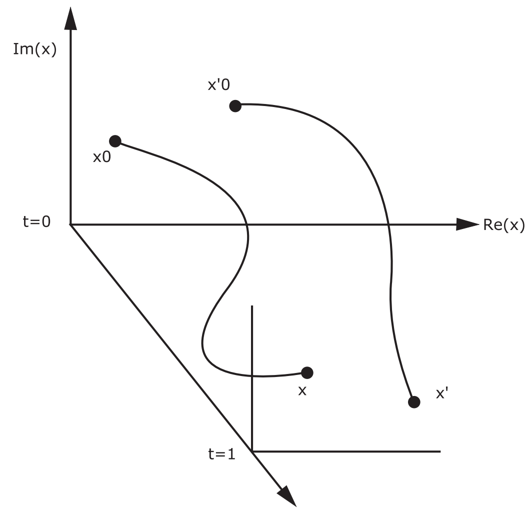

Figure 1. Two variable solution path by the continuation method [20]

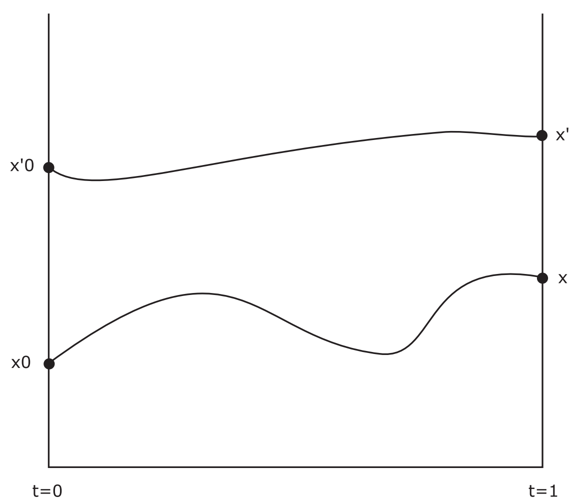

Figure 2. Path projection of the solutions by continuation method [20]

2.2. TENSEGRITY SYSTEMS

The term tensegrity was created by Richard Buckminster Fuller as a contraction of tensional and integrity.[8]. Although tensegrity structures are more that 60 years old, they started as an art manifestation and are currently used in many different fields like engineering, biomedicine and mathematics. Tensegrity structures are similar in appearance to the normal bar-joint structure, but in its static equilibrium there are presents forces within the structure that helps to balance and achieve its equilibrium, without this balance of forces the structures becomes unstable, this is the first problem in the design of the tensegrity structures and its called form-finding or shape finding [37]. The tensegrity structures have compressive parts and tensile parts. The compressive parts are simply rigid bodies, and the tensile parts simply strings.

Characteristics of a tensegrity structure:

- The structure is free-standing, without any support.
- The structural members are straight.
- There are only two different types of structural members: struts carrying compression and cables carrying tension.

In Art tensegrity structures are used because tensegrity components are very simple elements, often just straight lines. This type of beauty has appealed to sculptors like Kenneth Snelson [26] and architects like Buckminister Fuller [7]. Figure 3 shown a physical construction of a tensegrity structure.

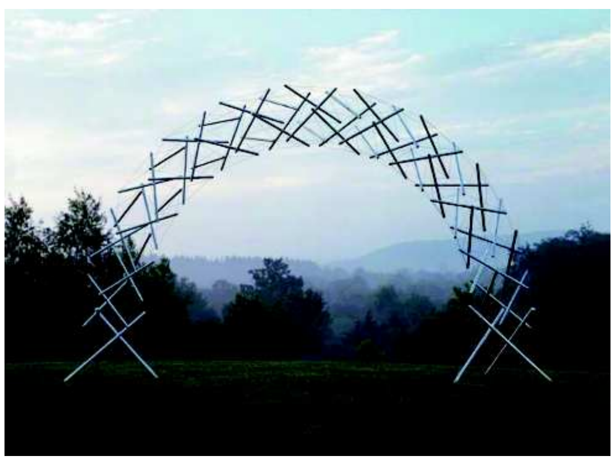

Figure 3. Tensegrity sculpture built by Kenneth Snelson [25]

In Nature, the bones and tendons of animals are connected in a manner that allows easy control of movement. Obviously, these structures evolved for control functions, where the bones provide compressive load-carrying capacity and the tendons provide the stabilizing tensions required in a given configuration. This structures are similar to a tensegrity structure and could be modeled by one.[25]

The simplest tensegrity structure is a single rigid body and a string. In the case of bars and strings, this is simply a prestressed bar. The next simplest, and first nontrivial tensegrity structure with bars and strings, is two bars and four strings, the next simplest tensegrity structure is shown in figure 4 is a fundamental three dimensional unit we will call a tensegrity prism. [25]

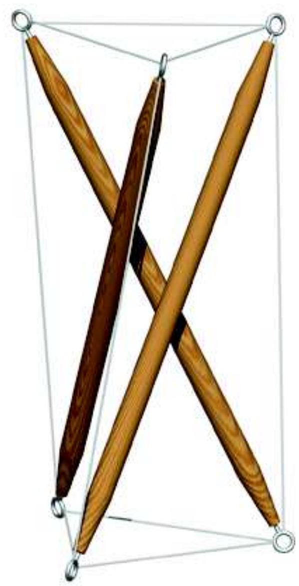

Figure 4. Tensegrity Prism [4]

Tensegrity structures have been studied and applied in a lot of fields, like civil engineering [9] , aerospace [6], wave energy harvesting [32], human anatomy [23], modeling the motion viruses [24] and aquaculture [11].

3. OBJECTIVES

3.1. GENERAL OBJECTIVE

Solve a static equilibrium problem for a tensegrity structure using the continuation method

3.2. SPECIFIC OBJECTIVES

- Define the tensegrity structure that will be used for the analysis
- Obtain the polynomial system that describe the equilibrium problem for the tensegrity structure
- Define the parameters and variation of the continuation method that will be used for solving the system
- Algorithm implementation of the continuation method for solving the polynomial system
- Design a graphical interface, for enter the parameters and shown the solution of the polynomial system

4. METHODOLOGY

The first step will be a research of the scientific sources; books and journals related with the solution of polynomial systems using the continuation method, the implementation of this method in different programming languages and the basics of tensegrity structures. The scientific data bases consulted will be Science Direct, ASME, Springer, IEEE

After do the compilation of the information, we will focus in understanding the continuation method, for this we will start with the implementation of small and basic polynomial systems, and increment the complexity until we can figure it out how the method works with general number of variables and equation and bigger degrees.

We will define the tensegrity structure, and do its static equilibrium analysis, obtaining the equations, then we will modify those equations to get a polynomial system.

After that we will implement an algorithm of the continuation method for solving the specific polynomial system, this implementation will be made in MATLAB.

When the algorithm is implemented, we will build a user interface, for other users to use the algorithm implementation and become a tool that can be used without the need of know how the continuation method works.

Write the manuscript and a paper to send to a scientific journal.

5. SCOPE, IMPACT AND CONTRIBUTION

The project looks for solving a static equilibrium problem, using an implementation of the continuation method, we are not looking to develop a new continuation method or a variation, we are looking for an implementation of an existing method in an algorithm that allow us to solve a static equilibrium problem focus in tensegrity structures.

We will do an analysis of a specific application, an analysis of the static equilibrium of a tensegrity structure, and develop a specific algorithm using the continuation method, we don´t pretend to formulate a new analysis of the static equilibrium of a tensegrity structure, neither an general purpose algorithm that works for all applications or all tensegrity structures.

The requirements for the algorithm implementation are being reliable and the result should correspond to the solution of the specific polynomial system, we are not looking for efficiency, optimization or enhancements of previous implemented algorithms.

We want to apply the continuation method to a problem, that has not been solve by this method.

The graphical interface will help to use the algorithm, in a easy way, without knowing its internal behavior or codification, it will not be error free and will need to follow the user manual for its correct performance.

The impact is academic and look forward to continue a investigation field of the research group: Grupo de Automática y Diseño A+D, of the Universidad Pontificia Bolivariana, Medellín.

After do the static equilibrium analysis of the tensegrity structures with the continuation method, we could find new solutions not approached before, that will be a new contribution to the analysis of tensegrity systems.

6. PRODUCTS

At the end of the research, we will submit:

- Thesis, with format provided for the UPB.
- Article for send to an indexed journal.
- Implemented Algorithm.
- Graphical Interface with user manual.
- Registered Software.

7. TIMETABLE

In Table 1 we present the timetable with the activities and the execution time of the thesis, it will have a 12 month duration, starting in July 2015 and ending in June 2016.

Table 1. Timetable

Activity	*July*	*August*	*September*	*October*	*November*	*December*	*January*	*February*	*March*	*April*	*May*	*June*
Bibliography Research	x	x	-	-	-	-	-	-	-	-	-	-
Choose Tensegrity Structure	-	x	x	x	-	-	-	-	-	-	-	-
Obtain Polynomial System	-	x	x	x	-	-	-	-	-	-	-	-
Apply the Continuation Method to the Polynomial system	-	-	-	x	x	-	-	-	-	-	-	-
Algorithm Implementation - Software	-	-	-		x	x	x	x	-	-	-	-
Design- Graphical Interface	-	-	-	-	-	-	x	x	x	-	-	-
Writing the manuscript	-	-	-	-	-	x	x	x	x	x	x	-
Review and corrections	x	x	x	x	x	x	x	x	x	x	x	-
thesis defense	-	-	-	-	-	-	-	-	-	-	x	x

8. BUDGET

In Table 2 we will present the necessary budget and resources for carrying it out the thesis

Table 2. Budget

	Participation in thousand Pesos		Require Disbursement	
Resources	*Student*	*University*	*Yes(New)*	*No(existent)*
Bibliography resources(Papers and books)	-	2000	-	x
Computer	1500	-	-	-
Software MATLAB	8000	-	-	x
Student Dedication $/-hour 20000	5200	-	-	x
Director Dedication $/-hour 80000	-	4160	-	x
Subtotal	14700	6160	-	-
incidentals (10%)	1470	616	-	-
Total -	22946			

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